Synchronous Chaotic Behaviour of a Response Oscillator with Chaotic Driving

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Abstract—Synchronization of chaotic self-excited oscillations and chaotic synchronous response are studied using chaotic electronic oscillators with unidirectional coupling.

1. INTRODUCTION

The phenomenon of synchronization of self-excited oscillators with chaotic behaviour has a high potential for applications in electronics, physics, biology and other fields. This phenomenon has been observed in studies of chaotic oscillators with mutual and unidirectional coupling, see for example [1–10]. In recent papers [11–13], stable regimes of synchronous chaotic response caused by chaotic driving were also demonstrated in several examples.

Attempts to understand the nature of the synchronization and the response, common features and differences of these phenomena comes from analogy with a classical problem of periodic synchronization and response in the van-der-Pol oscillator driven with an external force

\[ \ddot{x} - \mu(x^3 - x)\dot{x} + \omega^2 x = \eta \sin \omega_0 t \]  

(1)

where the parameters \( \eta, \mu \) and \( |\omega - \omega_0| \) are positive and much smaller than unity. If the parameter \( \Lambda \) has positive values, then equation (1) describes the phenomenon of forced synchronization of self-excited periodic oscillations. If \( \Lambda \) has negative value, then equation (1) describes the resonant response of the oscillator with nonlinear damping. Although this analysis refers to small parameter values, the nonlinear phenomena studied on the basis of this example have universal value to lead to understanding nonlinear behaviour in a large class of oscillatory systems.

In this paper we provide some new results when driving and response systems are described by the following equation

\[ \dddot{x} + A_3 \ddot{x} + A_2 \dot{x} + A_1 x + A_0 F(x) = 0. \]  

(2)

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Equation (2) was obtained as a model in a number of physical and electronic systems [6, 14, 15]. Its behaviour is controlled by parameters $A$, and the actual shape of the nonlinear function $F(x)$. In particular, the behaviour can be chaotic under certain conditions. In this paper we drive the system (2) by an electronic circuit (see Fig. 1) accounting for both driving, $G_1$, and response, $G_2$, systems. Specifically, we consider the case when a chaotic output signal $x_1(t)$ generated by $G_1$ acts as the external forcing for $G_2$.

2. DYNAMICS OF THE DRIVING CIRCUIT $G_1$

It is known [6, 14, 15] that the circuit $G_1$ can operate as autonomous, self-excited oscillator when $S_{11}$ is switched on. Let $x_1(t)$ and $z_1(t)$ be the voltages at capacitors $C_{12}$ and $C_{11}$, respectively. The voltage $x_1(t)$ is used as a driving signal taken from terminal $b_1$ and applied to terminal $a_2$ of the response oscillator. The nonlinearity $F(x)$ is produced by the nonlinear amplifier 1 (for details see Ref. [10]). The low-frequency filter $R_{12}C_{11}$ and the resonant circuit $L_1R_{12}C_{12}$ serve as a feedback element. The dynamics of the circuit can be described by a three-variable system of differential equations

$$
\begin{align*}
\dot{x}_1 &= v_1, \\
\dot{y}_1 &= -x_1 - \delta_1 y_1 + z_1, \\
\dot{z}_1 &= \gamma_1 [\alpha_1 F(x_1) - z_1] - \sigma_1 y_1,
\end{align*}
$$

with

$$
\begin{align*}
y_1(t) &= i_1 \sqrt{\left( \frac{L_1}{C_{12}} \right)}, \\
\tau_{\text{new}} &= \frac{t}{\sqrt{(L_1C_{12})}}, \\
\gamma_1 &= \frac{\sqrt{(L_1C_{12})}}{R_{12}C_{11}}, \\
\delta_1 &= R_{12} \sqrt{\left( \frac{C_{12}}{L_1} \right)}, \\
\sigma_1 &= \frac{C_{12}}{C_{11}},
\end{align*}
$$

Fig. 1. Block scheme of the electronic circuits modelling the driving chaotic oscillator $G_1$ and the response oscillator $G_2$. 1 denotes the nonlinear amplifier, 2 and 3 are subtractor and adder, respectively, $U_{21}(t) = \epsilon(x_1(t) - x_2(t))$, $U_{22} = x_2(t) + U_{21}(t)$. 

The control parameter \( \alpha \) characterizes the gain of the nonlinear amplifier around \( \alpha = 0 \). For simplicity, in the text below we will use the letter \( \tau \) instead of \( t_{new} \), and we shall disregard subscripts in the parameters unless we strictly need them.

The system (3) can be rewritten in the form of equation (2), by using the change of variables and parameters
\[
\begin{align*}
x &= x_1, \\
A_3 &= \delta_1 + \gamma_1, \\
A_2 &= 1 + \sigma_1 + \delta_1 \gamma_1, \\
A_1 &= \gamma_1 \\
A_0 &= -\gamma_1 \alpha_1.
\end{align*}
\]
Thus our circuit with different shapes of \( F(x) \) can serve as an electronic model for chaotic oscillators in a wide range of physical problems.

If \( \alpha \) is higher than \( \alpha_0 = -0.227 \) but less than 1, then all trajectories starting at different points of phase space of the system (3) approach a single stable fixed point at origin \( O_0 \), because all other parameters can have only positive values. In this parameter region the circuit can produce only damped oscillations, and the system behaves as a nonlinear oscillator with damping. When \( \alpha \) becomes higher than 1 but remains less than \( \alpha_H = 1 + [(\delta + \gamma)(1 + \sigma + \delta \gamma)]/2\gamma \), the fixed point \( O_0 \) is no longer stable and the oscillator operates in a multistable regime with two additional stable stationary points, \( O_L(-\sqrt{[(\alpha - 1)/\alpha]}) \), \( O_R(\sqrt{[(\alpha - 1)/\alpha]}) \), and \( O_{R'}(\sqrt{[(\alpha - 1)/\alpha]}) \). Transition through the critical value \( \alpha_H \) is accompanied by a supercritical Andronov-Hopf bifurcation. Then the fixed points \( O_L \) and \( O_R \) lose stability and two stable limit cycles \( P_L \) and \( P_R \) appear in the phase space of the system. These limit cycles are topologically similar and can be superimposed under the transformation \( x \to -x, y \to -y, z \to -z \). Thus at the threshold value \( \alpha_H \) the circuit becomes a self-excited periodic oscillator operating in a multistable regime.

For higher values of \( \alpha \) with fixed parameters \( \delta = 0.43 \) and \( \sigma = 0.72 \), we have also explored the dynamics experimentally and with numerical simulations. It appears that for small enough values of \( \gamma \), when \( \alpha \) increases a period doubling bifurcation sequence of Feigenbaum type from \( P_L \) and \( P_R \) leads to the appearance of two strange attractors \( SA_L \) and \( SA_R \). These attractors correspond to two types of chaotic oscillations. Then \( SA_L \) and \( SA_R \) merge into a single symmetric strange attractor \( SA \) when \( \alpha \) is higher than some critical value. The dependence of maximum nonzero Lyapunov exponents of attractors on the parameter \( \alpha \) with \( \gamma = 0.1 \) and projections of \( SA_L \), \( SA_R \) and \( SA \) onto the plane \((x, z)\) are shown in Fig. 2. In an experimental study of the circuit G1 [14], it was shown that the chaotic behaviour in some appropriate region of the parameters is connected with bifurcations of homoclinic trajectories of saddle-focus \( O_0 \) with positive saddle-focus value [16].

### 3. Dynamics of the Response System G2

As a response system we employ a circuit similar to the driving oscillator, but the voltage \( x_2 \) at the capacitor \( C_{22} \) is not applied directly to the input of the nonlinear amplifier 1 (cf. G2 to G1 in Fig. 1). In G2 the voltage is led to circuits 2 and 3 when \( S_{21} \) is switched on. The circuit 2 subtracts \( x_2(t) \) from the signal \( x_1(t) \) applied to the input \( a2 \) from the output of G1. The result of this subtracting \( U_{21} = \epsilon(x_1(t) - x_2(t)) \) adds with \( x_2(t) \) by means of the circuit 3.

For simplicity we restrict the problem to the case when the corresponding parameters \( C_{12} \)
and $L$ in $G_1$ and $G_2$ have equal values. Thus $G_2$ can be described by the system of equations

$$\begin{align*}
\dot{x}_2 &= y_2, \\
\dot{y}_2 &= -x_2 - \delta y_2 + z_2, \\
\dot{z}_2 &= \gamma_2 [\alpha_2 F(x_2 - \epsilon(x_2 - x_1(t))) - z_2] - \sigma_2 y_2,
\end{align*}$$

where $\epsilon$ is the gain of circuit 2. The parameter $\epsilon$ points out the level of unidirectional coupling between $G_1$ and $G_2$ and can be used as a control parameter. Note that if $\epsilon = 0$, system (5) becomes equivalent to (3).

In order to consider the dynamics of $G_2$ separately we switch off $S_{12}$. In this case the model of $G_2$ is given by (5) with $x_1 = 0$. It is easy to show that, using new variables $x_{\text{new}} = (1 - \epsilon)x_2$, $y_{\text{new}} = (1 - \epsilon)y_2$ and $z_{\text{new}} = (1 - \epsilon)z_2$, we reobtain equations (3) with a normalized parameter $\alpha_{\text{norm}} = \alpha_2 (1 - \epsilon)$. Thus, the response system possesses individual dynamics identical to the driving system. The coupling parameter $\epsilon$ can change the dynamics of the response system by introducing some shift along the parameter $\alpha$. For values of $\epsilon$ such that

$$\epsilon > \epsilon_H = 1 - (\alpha_H/\alpha_2),$$

the response system is not a self-excited oscillator. Variation of $\epsilon$ from $\infty$ down to 0 reproduces the dynamics displayed by the driving oscillator when $\alpha_1$ is varied from $-\infty$ up.
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This enables us to use the coupling parameter in a manner similar to the role played by the parameter \( \Lambda \) in system (1).

4. SYNCHRONIZATION OF CHAOS AND SYNCHRONOUS CHAOTIC RESPONSE FOR OSCILLATORS WITH IDENTICAL PARAMETERS

Let us consider the dynamics of the coupled oscillators when the feedback parameters and \( \alpha \) have identical values. In this case there exists an invariant manifold

\[
\begin{align*}
x_1 &= x_2, \\
y_1 &= y_2, \\
z_1 &= z_2
\end{align*}
\]

in the six-dimensional phase space of the system of equations (3) and (5) taken together.

Trajectories of the system (3) and (5), located in this three-dimensional manifold, correspond to synchronous chaotic oscillations when the parameters of Gl are in a region of chaotic behaviour. If these trajectories belong to the attractor, then G2 will operate in a stable regime of synchronous reproduction of the chaotic oscillations generated by G1.

It is essential that the manifold exists for all values of the parameter \( \epsilon \). This enables us to investigate the stability of the chaotic limit set located in the manifold as a function of the coupling parameter. The Lyapunov exponent spectrum (LES) of the limit set can be divided into two parts \( LES_\parallel \) and \( LES_\perp \), respectively along and orthogonal to the manifold. Thus \( LES_\parallel \) contains three Lyapunov exponents corresponding to the evolution of perturbations tangent to the manifold. This part of the LES comes from the driving system G1. The components of \( LES_\perp \) are given by the evolution of perturbations transverse to the manifold. This part of the LES is equivalent to the conditioned Lyapunov exponent introduced in Ref. [11].

The linear approximation used to calculate \( LES_\perp \) can be written in the form

\[
\ddot{\xi} + (\delta + \gamma) \dot{\xi} + (1 + \sigma + \delta \gamma) \dot{\xi} + \gamma \xi + \gamma \alpha (1 - \epsilon) \Psi(t) \dot{\xi} = 0,
\]

where \( \Psi(t) = (dF(x)/dx)_{t=x(t)} \).

If the coupling parameter \( \epsilon \) is taken equal to 0, then the system (8) will give us \( LES_\parallel \). Note that, owing to the time-dependence in Equation (8), the stability analysis of the point \((0, 0, 0)\) defined by (7) demands consideration of the initial-value problem which leads to the computation of the Lyapunov exponents along the corresponding trajectory, \( x_i(t) \).

The dependence of the two largest Lyapunov exponents of \( LES_\perp \) on \( \epsilon \) is presented in Fig. 3(a), with the parameters of oscillators taken the same as in the case shown in Fig. 2(c). It clearly appears that there is a region of coupling values where all components of \( LES_\perp \) are negative, hence the chaotic limit set located in the manifold is an attractor.

Note that if \( \epsilon = 1 \) the system (3) and (5) describes the case similar to the examples considered in [11–13]. In our case, with \( \epsilon = 1 \) the system (8) becomes autonomous and the components of \( LES_\parallel \) are given by the eigenvalues of the linear system. By using the Routh–Hurwitz criterion, it is easy to show that all eigenvalues of the system (8) have negative real part, thus showing the manifold stability. In Section 3 it has been shown that, when \( \epsilon = 1 \) and without external forcing \( x_i(t) \), the oscillator G2 is a damped oscillator. According to the analogy with the forced van-der-Pol oscillator (1), we conclude that in the considered case the synchronous behaviour of G2 corresponds to a synchronous chaotic response. When \( \epsilon \) decreases, we have a response behaviour down to the critical value \( \epsilon_{H} \) given by (6). For the chosen parameter values of the oscillators \( \epsilon_{H} = 0.801 \), and this value lies inside the manifold stability region (see Fig. 3a). Typical experimental results obtained with the electronic circuits are displayed in Fig. 4. Figures 4(a) and (c) correspond to the
Fig. 3. (a) Dependence of the two largest Lyapunov exponents of $\text{LES}_1$ on $\epsilon$, calculated using (8). Displayed here is the component of the Lyapunov exponent orthogonal to the manifold (7) and positive value indicates instability of the $(0,0,0)$. (b) Dependence of the four largest Lyapunov exponents of $\epsilon$ for attractors of the complete six-equation system (3) and (5) together for $\alpha = 32$, $\delta = 0.43$, $\sigma = 0.72$, and $\gamma = 0.1$. In the region of synchronization $0.4 < \epsilon < 1.1$ the difference between (b) and (a) is due to algorithmic differences (six versus three equations).

Fig. 4. Experimental illustration of the synchronous chaotic response as it appears on the display of the oscilloscope: (a) and (c) are the $(x, z)$ projections of typical attractors in $G_1$ and $G_2$, respectively, and Fig. 4(b) depicts the synchronization: it shows the $(x_1, x_2)$ manifold when both oscillators are synchronized.

$(x, z)$ projections of chaotic attractors in $G_1$ and $G_2$, respectively and Fig. 4(b) depicts the synchronization: it shows the $(x_1, x_2)$ manifold when both oscillators are synchronized.

If $\epsilon$ is taken smaller than $\epsilon_H$, but in the region of the manifold stability, then $G_2$ experiences a genuine synchronization phenomenon because, without driving, when $S_{12}$ is switched-off, $G_2$ behaves as a self-excited oscillator. For example, when $\epsilon = 0.5$, the
individual dynamics of G2 corresponds to chaotic oscillations. In phase space these oscillations look like the strange attractor displayed in Fig. 2(b) but magnified twice (see Section 3). When G1 and G2 are not connected they produce uncorrelated chaotic oscillations with different features (see Fig. 2c and 2b). If \( S_{12} \) is switched-on and the chaotic signal \( x_1(t) \) is applied to G2, then the chaotic oscillation will be synchronized by the chaotic driving because the value \( \epsilon = 0.5 \) is in the region of the manifold stability. Numerical simulations show that, for the chosen values of the parameters, the synchronized chaotic oscillations is a single stable regime.

When \( \epsilon \) is decreased below or increased above the limits of the stability region of the manifold, the oscillatory regime of the full system composed of equations (3) and (5) is characterized by uncorrelated oscillations in G1 and G2. This behaviour is shown in Fig. 3(a) for the linearized system (8). The existence of one positive \( \text{LES} \) indicates instability of the manifold in the corresponding two regions. In Fig. 3(b) the four largest \( \text{LES} \) of the full system (3) and (5) are shown. For values of \( \epsilon < 0.4 \), the oscillatory regime is hyperchaotic (two positive \( \text{LES} \)), while it is simply chaotic in region \( \epsilon > 1.1 \). Both regions are outside the manifold. Figure 3(b) is not necessary. However, it helps to make this paper self-contained and to cross-check that indeed \( x_1(t) \) is chaotic as already shown in a previous publication, Rulkov et al. [10]. Thus in Fig. 3(a) we display the amplification rate associated to the component of the Lyapunov vector orthogonal to the manifold, whereas Fig. 3(b) shows the full Lyapunov exponent in modulus. Obviously the modulus is larger than or equal to the amplification rate associated to any of the components of the Lyapunov vector. As the \( x_1 \) signal is chaotic, there is indeed one nonvanishing component along the manifold. Note also that, for instance, for \( \epsilon < 0.4 \) the positive value of exponent displayed in Fig. 3(a) is always smaller, or at most equal to, the sum of the two positive exponents shown in Fig. 3(b).

5. SYNCHRONOUS CHAOTIC OSCILLATIONS WITH NONIDENTICAL PARAMETERS OF THE OSCILLATORS

The spread in the parameters of the oscillators leads to distortion of the manifold (7) in phase space of the system (3) and (5) and, as a result, the synchronous chaotic oscillations are no longer uniform oscillations. In order to study new features of the synchronous chaotic behaviour of the oscillators, let us consider the simplest case \( \epsilon = 1 \). This assumption corresponds to the case when \( S_{11} \) is switched off (see Fig. 1). In this case, the driving signal \( x_1(t) \) comes through the nonlinear amplifier 1 and then the modified signal \( \alpha_2 F(x_1(t)) \) is applied to the linear filter, which is no longer a feedback element.

Trajectories in the phase subspace \((x_2, y_2, z_2)\) are given now by the solutions of the linear nonautonomous system (5). Because of the linearity of the system any solution can be obtained as the particular solution of the nonhomogeneous system plus the complementary solution of the homogeneous system which corresponds to the system (5) with \( \epsilon = 1 \). The complementary solution decays with time, because the eigenvalues of the linear system have negative real parts. Thus, if \( x_1(t) \) is a solution on the attractor in the phase space of the driving oscillator then the motions along attractor in the phase subspace \((x_2, y_2, z_2)\) will be given only by the particular solution. This solution can be written in the form

\[
\begin{align*}
x_2(t) &= \hat{K}_{x_2}(p)\hat{K}_{z_2}^{-1}(p)\alpha_2 F(x_1(t)), \\
y_2(t) &= \hat{K}_{y_2}(p)\hat{K}_{z_2}^{-1}(p)\alpha_2 F(x_1(t)), \\
z_2(t) &= \hat{K}_{z_2}(p)\hat{K}_{z_2}^{-1}(p)\alpha_2 F(x_1(t)),
\end{align*}
\]  

(9)
where \( p = d/dt \), the linear differential operators are given by following expressions:

\[
\begin{align*}
\hat{K}_i(p) &= p^3 + (\delta_i + \gamma_i)p^2 + (1 + \sigma_i + \gamma_i\delta_i)p + \gamma_i, \\
\hat{K}_{ii}(p) &= \gamma_i, \\
\hat{K}_{yi}(p) &= \gamma_ip, \\
\hat{K}_{zi}(p) &= \gamma_i(1 + p(p + \delta_i)).
\end{align*}
\]

Let us imagine that we have recorded the signal \( x_i(t) = x_i(t) \) which corresponds to trajectory located in the strange attractor in the phase subspace \( (x_i, y_i, z_i) \). Then we switch off \( S_{1\downarrow} \) and apply the signal \( x_i(t) \) to the terminal \( a_1 \) (see \( G_1 \) Fig. 1). Transformations of the signal by the nonlinear amplifier 1 and in the linear filter are similar to the case considered above for \( G_2 \). Oscillations in \( G_1 \) will be described by the following equations:

\[
\begin{align*}
\dot{x}_1 &= y_1, \\
\dot{y}_1 &= -x_1 - \delta_1y_1 + z_1, \\
\dot{z}_1 &= -\gamma_1z_1 - \sigma_1y_1 + \gamma_1a_1F(x_i(t)).
\end{align*}
\]  

After some transient time, the voltage at the capacitor \( C_{1\downarrow} \) will be equal to the voltage \( x_i(t) \) because it is a solution of the system (3). The particular solution of the nonhomogeneous linear system (11) is of the form given by (9), where subscripts 2 should be replaced by 1. According to the changes considered in the circuits, the equations (5), (9), (10) and (11) enable us to write down the following connection between the trajectories in the phase subspaces \( (x_2, y_2, z_2) \) and \( (x_1, y_1, z_1) \):

\[
\begin{align*}
x_2(t) &= \hat{K}_{x_2x_1}(p)x_1(t), \\
y_2(t) &= \hat{K}_{y_2y_1}(p)y_1(t), \\
z_2(t) &= \hat{K}_{z_2z_1}(p)z_1(t),
\end{align*}
\]

where

\[
\hat{K}_{ul,uy}(p) = \frac{\hat{K}_{u}(p)\hat{K}_y(p)}{\hat{K}_{y}(p)\hat{K}_u(p)}
\]

are integrodifferential operators.

Owing to the linearity of the equations (12), we can make use of the Fourier transform and consider the connections between complex amplitude spectra of the realizations. Then we get

\[
\begin{align*}
\hat{S}_{x_2}(i\omega) &= \hat{K}_{x_2x_1}(i\omega)\hat{S}_{x_1}(i\omega), \\
\hat{S}_{y_2}(i\omega) &= \hat{K}_{y_2y_1}(i\omega)\hat{S}_{y_1}(i\omega), \\
\hat{S}_{z_2}(i\omega) &= \hat{K}_{z_2z_1}(i\omega)\hat{S}_{z_1}(i\omega),
\end{align*}
\]

where \( \hat{K}_{ul,uy}(i\omega) \) is a complex function obtained form \( \hat{K}_{ul,uy}(p) \) with \( p = i\omega \).

These expressions show the existence of hard linking between the spectrum distribution of realizations of chaotic oscillations in \( G_1 \) and \( G_2 \). Study of phase distributions of the continuous spectra of the chaotic realizations in \( G_1 \) and \( G_2 \) demonstrates that every harmonic of the response-oscillation spectrum is locked by the harmonic with the same frequency contained in the driving signal. This property of coupled oscillators can also be employed as a definition of synchronous behaviour. Thus if one knows the trajectory on the chaotic attractor in the phase space \( (x_1, y_1, z_1) \) of the driving system then projection of
the chaotic trajectory located on the attractor in the phase subspace \((x_2, y_2, z_2)\) of the response system is single-valued, and vice versa.

The property shown in (12) and (13) is in accordance with the more rigorous mathematical definition of the synchronous chaotic oscillations given in [1]. Applied to our particular case this definition can be presented in the following way. Let us assume that \(G_1\) and \(G_2\) have attractors \(A_1\) and \(A_2\), respectively, with \(S_{12}\) switched off. The synchronous oscillations occur for some value of coupling between \(G_1\) and \(G_2\) if the attractor \(A_3^*\) in the six-dimensional phase space of the system (3), (5) is such that: (1) its projections \(\pi_1(A_3)\) and \(\pi_2(A_3)\) onto phase subspaces \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) are homeomorphic (note that \(\pi_1(A_3)\) equals \(A_1\) because \(G_2\) has no influence on \(G_1\)); (2) there exists a homeomorphic reflection \(g: \pi_1(A_3) \rightarrow \pi_2(A_3)\) which satisfies the following properties: (a) \(g\) is differentiable in the vicinity \(\pi_1(A_3)\), (b) every point along any trajectory on the attractor \(A_3\) has single-valued projections onto the phase subspaces.

It should be noted that application of this definition in experiments and numerical simulations with chaotic oscillators is not easy unless a manifold like that considered in Section 4 is used. Taking into account the complicated structure of the strange attractors, it becomes clear that the difficulties lie in proving the first requirement because of the limited accuracy of measurements and numerical simulations. For this reason, applications of the definition of the synchronous chaotic oscillations by means of spectra connections (13) may be preferable in experimental studies. Mention should be made that some hypotheses of mutual correlation of harmonic phases in continuously distributed spectra of realizations were employed in earlier numerical simulations [5, 7].

6. CONCLUSIONS

It has been demonstrated, by using unidirectionally coupled nonlinear circuits with chaotic behaviour, that synchronous chaotic oscillations can arise as the result of either forced synchronization or response phenomena. The main difference between these two phenomena rests on individual dynamics of the response system, as happens with the periodically forced van-der-Pol oscillator (1).

The study of synchronous chaotic response properties presented in Section 5 enables us to conclude that any linear oscillator which has an exponentially stable stationary state will oscillate in the regime of synchronous chaotic response with chaotic external forcing. In this regime such characteristics as KS-entropy and dimensions of the chaotic attractors, calculated from realizations of driving and response oscillators, are the same in both oscillators.

It should be noted that synchronized chaotic oscillations in a chain of coupled chaotic oscillators can lead to the appearance of travelling chaotic stationary waves. In order to clarify such kind of nonlinear behaviour, let us consider the chain depicted in Fig. 5, where \(G_1\) is the driving chaotic oscillator, \(G_2, G_3\) and so on are the response systems like \(G_2\) in Fig. 1. The oscillators are placed along the space coordinate \(s\) with spacing \(L\). Let unidirectional coupling between the oscillators contain the time delay \(\tau\). Consider the case when the parameters of the oscillators are \(\alpha_i = 32, \gamma_i = 0.1, \delta_i = 0.43, \sigma_i = 0.72\) and \(e_i = 0.5\) where \(i = 1, \ldots, \infty\). When \(S_{12}\) is switched on, the chaotic driving signal \(x_1(t - \tau)\) will synchronize the response oscillator \(G_2\), because the coupling parameter \(c\) is inside the region of manifold stability (see Fig. 3). As a result of the synchronization, \(G_2\) will generate a signal \(x_2(t) = x_1(t - \tau)\). Then \(x_1(t - \tau)\) is applied to \(G_3\) as the driving chaotic signal and will synchronize \(G_3\). As a result of the synchronization, the oscillator \(G_n\)
Fig. 5. Block diagram of the chain of unidirectionally coupled chaotic oscillators. The driving oscillator G1 corresponds to G1 shown in Fig. 1. The chain of response chaotic oscillators Gi, i > 1 contains the response oscillators which are equivalent to G2 in Fig. 1.

reproduces the chaotic driving signal with a time delay $n\tau = (s/L)\tau$ and the solution in the chain can be presented in the form

$$x_i(t - \frac{\tau}{L} s).$$

This solution is a chaotic wave shaped by the solution of the driving oscillator and it travels along the chain with constant velocity $v_\tau = L/\tau$. The recent results of stability study of similar waves observed in chain of coupled mappings [17] enable us to conclude that the wave can be stable, at least inside the domain $0 < s < NL$, where the number $N$ depends on the actual noise level in the experiment, or on the accuracy of the numerical simulations, and can be quite large. When $S_{12}$ is switched off and, as a result, the driving signal is not applied to G2, then the individual dynamics of the oscillators remain chaotic (see Section 3), but the oscillations are no longer synchronized and the chain shows turbulent behaviour.

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