Robustness and Stability of Synchronized Chaos: An Illustrative Model
Mikhail M. Sushchik, Jr., Nikolai F. Rulkov, and Henry D. I. Abarbanel

Abstract—Synchronization of two chaotic systems is not guaranteed by having only negative conditional or transverse Lyapunov exponents. If there are transversally unstable periodic orbits or fixed points embedded in the chaotic set of synchronized motions, the presence of even very small disturbances from noise or inaccuracies from parameter mismatch can cause synchronization to break down and lead to substantial amplitude excursions from the synchronized state. Using an example of coupled one dimensional chaotic maps we discuss the conditions required for robust synchronization and study a mechanism that is responsible for the failure of negative conditional Lyapunov exponents to determine the conditions for robust synchronization.

Index Terms—Chaos, stability, synchronization.

I. INTRODUCTION

Since the first observations of synchronized chaos [1]–[3], this behavior has been the subject of a substantial number of investigations. This arises both because of the intrinsic interest in the idea of synchronization between chaotic, irregular motions and because of potential applications. Among the possible applications are secure communications [3]–[8] and control of systems with chaotic behavior [9]–[11]. There have also appeared several papers [12]–[14] which establish the clear connection between synchronization and modern nonlinear control theory [15]. For the applications of synchronized chaos it is important to establish criteria that would determine the conditions, for example, the value of the coupling strength, that guarantee that two systems would demonstrate steady, undisrupted synchronization. In practice one frequently uses the negativity of all transversal or conditional Lyapunov exponents, so that oscillations transverse to the synchronization manifold should contract back to that manifold. This method is straightforward and universal, however, in practice it appears not to be a sufficient condition for stable operation of the system in the regime of synchronized chaos [16]–[22].

When the conventional transversal Lyapunov exponents are evaluated, they are computed along a chaotic trajectory in the invariant chaotic set of synchronized motions. The negativity of these exponents indicates that in the neighborhood of the chaotic set of synchronized motions there is a set of initial conditions of nonzero measure such that trajectories originating at these points are attracted to the chaotic set of synchronized motions. At the same time, the negativity of global transversal Lyapunov exponents is not sufficient to guarantee the linear stability of the chaotic set as a whole. This is because, besides trajectories coming to the chaotic set of synchronized motions, there may still exist trajectories departing from the chaotic set. The mathematical analysis of the problem of transversal stability for chaotic sets shows that to study the transversal stability of an invariant chaotic set, one must consider Lyapunov exponents computed for all invariant measures defined on this set (see for example [23]). When this stability analysis is applied to the problem of synchronization of chaos, it determines when all initial conditions in a small region fully containing the chaotic set of synchronized motions are attracted toward it.

Stated plainly, this means that within the set of synchronized trajectories left invariant by the dynamics there may be both chaotic trajectories where the transversal Lyapunov exponents are negative and other trajectories that have positive transversal exponents. In the usual analysis, the Lyapunov exponents are computed along the chaotic trajectory. However, when the system evolves in the vicinity of a transversally unstable orbit, noise or other disturbances can drive the system away from the manifold of synchronized motions. The presence of these “embedded” unstable orbits means that one must locate them, or at least the most unstable among them, and make sure that the coupling between the systems to be synchronized is sufficiently large to overcome these additional, unwanted instabilities.

Using the example of coupled maps, we shall show that only transversal stability of the whole chaotic set of synchronized motions can guarantee the robustness of synchronized chaotic oscillations. When this chaotic set is unstable and transversal Lyapunov exponents for some invariant measures defined on it are positive, the introduction of arbitrarily small disturbances into the system, for instance noise in the coupling channel, can trigger finite size outbursts of unsynchronized behavior. We shall show that this can occur even when the largest transversal Lyapunov exponent computed along a chaotic trajectory is negative. Such behavior is explained by the presence of transversally unstable fixed points and periodic orbits embedded in the chaotic set of synchronized oscillations. Thus, we show that the robustness of synchronized chaos is determined both by the global transversal Lyapunov exponents computed...
for chaotic trajectories and by the transversal characteristic exponents of the fixed points and periodic orbits contained in the chaotic set of synchronized motions. Identifying those additional regions of transversal instability allows us, in our example, to set a criterion for the onset of stable synchronization between the systems.

II. ROBUSTNESS AND STABILITY OF SYNCHRONIZED CHAOS IN MAPS

A. The Coupled Maps; Conventional Transversal Lyapunov Exponents Analysis

Our example system in this paper is comprised of the two coupled maps

\[ x_{n+1} = f(x_n) \]
\[ y_{n+1} = f(y_n) + g[f(x_n) - f(y_n)] \]

where

\[ f(x) = \frac{\alpha x (1 - x^2)}{1 + \beta x^2} \]

and \( g \) is a scalar coupling parameter \( 0 < g < 1 \). We choose the parameters \( \alpha = 5 \) and \( \beta = 2 \) which gives rise to chaotic motions of the \( x_n \). The first map \( x \to f(x) \) is autonomous and drives in a unidirectional fashion the dynamics of \( y_n \). This is the kind of setup appropriate for the communications and control applications mentioned at the beginning. Synchronized motions of these coupled maps is just \( x_n = y_n \) which is clearly a solution of the system.

The chaotic attractor of the map is shown in Fig. 1. This simple example has been selected as it is very easy to see the mechanism that leads to instability of the regime of synchronized chaos even when the transversal Lyapunov exponent is negative.

Since the coupled maps are each one-dimensional, it is very easy to compute the global transversal instability allows us, in our example, to set a criterion for the onset of stable synchronization between the systems.

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B. The Transversal Stability of the Chaotic Set of Synchronized Motions

To address this question, we performed the following numerical tests: i) We iterated maps (1), and (2) and recorded the maximum deviation \( |x_n - y_n| \) from the synchronization manifold as a function of the coupling strength \( g \) and ii) small amounts of noise were introduced into the response map as

\[ y_{n+1} = f(y_n) + g[f(x_n + \eta_n) - f(y_n)] \]

\( \eta_n \) is white noise uniformly distributed within the interval \([-\eta; \eta]\). The maximum deviation was evaluated as we varied the amplitude of the noise, \( \eta \), over a small range near zero; \( 0 \leq \eta \leq 0.04 \). The results are summarized in Fig. 3.

One can see that even noise of very small amplitude has a very dramatic effect on synchronization. Without noise, as predicted by the transverse Lyapunov exponent analysis, the
maps synchronize when \( g > g_0 = 0.58 \). One sees an abrupt onset of synchronization. However, when noise is present, the transition to synchronization has a smoother character and the maps remain unsynchronized until higher values of the coupling. This behavior does not depend strongly on the particular kind of disturbance introduced into the system. As a check, we conducted the same numerical tests with noise added both in driving and in response maps. The results were not qualitatively different from those shown in Fig. 3.

The situation with noise more accurately describes the practical situation as noise is always present in physical systems. Our example, thus, shows that although \( \lambda_{\perp} \) is negative and there is a set of initial conditions of nonzero measure attracted toward the chaotic set of synchronized motions, this by itself does not ensure the stable evolution of coupled maps in the regime of synchronization.

Let us now find the value of \( g \) at which the whole chaotic set of synchronized motions becomes transversally stable. Following the work of Ashwin et al., we take into account contributions from other invariant measures defined on the chaotic set of synchronized motions. Thus, we define a whole class of transversal exponents \( \lambda_n \):

\[
\lambda_n = \log |1 - g| + \left\langle \log \left| \frac{df(\xi)}{d\xi} \right| \right\rangle_{\nu_n} \tag{10}
\]

where \( \{\nu_n\} \) is the set of invariant measures we consider. In our example some of these exponents remain positive up to \( g_0 = 0.8 > g_0 \).

Let us consider the set of fixed points and periodic orbits embedded into the chaotic set of synchronized motions. There is an invariant measure \( \rho_{p, k}(x) \) that corresponds to each of these fixed points and periodic orbits:

\[
\rho_{p, k}(x) = \frac{1}{p} \sum_{i=1}^{p} \delta[x - f^i(x_{p, k})] \tag{11}
\]

where \( p \) stands for the period of one of these orbits, \( k \) identifies the periodic orbits of a given period and \( x_{p, k} \) is a point on a periodic orbit. Using these measures in (10) transforms the last term into the characteristic exponents of fixed points and periodic orbits associated with the driving map. We computed these exponents for all fixed points \( p = 1 \), and periodic orbits of order up to \( p = 10 \). The results are shown in Fig. 4. We see that the characteristic exponents for the fixed point \( z_{n} = 0 \) and for many periodic orbits are larger than the Lyapunov exponent of the chaotic attractor in the driving map (\( \lambda = 0.87 \)). Therefore, stronger coupling will be necessary to ensure the transversal stability of the corresponding fixed points and periodic orbits in the coupled system (1), (2). The fixed point at the origin is characterized by the greatest transversal instability. This is due to the fact that the derivative of the mapping function \( f(x) \) is maximum at that point.

Fig. 5 shows \( \lambda_n \) as a function of the coupling strength \( g \) for a few of the most transversally unstable fixed points and periodic orbits with \( p \leq 10 \). One can see that all \( \lambda_n \)
become negative when \( g > 0.8 \). For \( g > 0.8 \) all fixed points and all periodic orbits in the coupled system become transversally stable. It is worth noticing that the upper bound for the characteristic exponents in the driving map (see Fig. 5) approaches the characteristic exponent of \( x = 0 \) from below as the order of orbits increases. The higher values of the characteristic exponents correspond to the orbits that spend more time in the neighborhood of the fixed point \((0, 0)\). Therefore, as \( g \) decreases below the critical value, an infinite number of high-order periodic orbits become transversally unstable.

Although in our analysis we did not really consider all invariant measures, it is reasonable to assume that the invariant measure corresponding to the least transversally stable fixed point \((0, 0)\) will determine the overall transversal stability of the chaotic set of synchronized motions, which agrees with the observations in [24]. Thus we expect that for \( g > \bar{g}_0 = 0.8 \) the entire chaotic set of synchronized motions will be transversally stable. In Appendix A, we show how the linear transversal stability of this set can be proven by using the ideas behind the method of contraction mappings.

C. The Effect of Noise on Synchronization of Chaos

In the previous sections, we considered two approaches to the analysis of transversal stability of synchronized motions. The first approach was based on the calculation of transversal Lyapunov exponents which only characterize the transversal stability of the chaotic trajectory in the synchronization manifold. In the second approach, the stability analysis is done for the entire chaotic set of synchronized motions, including embedded fixed points and periodic orbits. It was shown by numeric simulation that in the presence of arbitrarily small (but nonzero) noise the first approach cannot be used for calculation of the critical value of coupling at which robust synchronization sets in. We shall now discuss why the transversal stability of fixed points and periodic orbits in the chaotic set of synchronized motions is so important for robustness of synchronized chaos.

Let us consider the mechanism of noise amplification during the systems evolution in the neighborhood of the manifold of synchronized motions. In the presence of noise, the evolution of \( z_n = y_n - x_n \) is governed by the following equation:

\[
\begin{align*}
  z_{n+1} &= (1-g)f(y_n) - f(x_n) + g f(x_n + \eta_n) \\
  &= (1-g)f(x_n + z_n) - f(x_n) + g f(x_n + \eta_n).
\end{align*}
\]

The evolution of small perturbations away from the synchronization manifold is obtained by linearizing this near the solution \( z_n = 0, \eta_n = 0 \)

\[
z_{n+1} = \frac{df(x)}{d\xi} \bigg|_{\xi=\epsilon_n} [(1-g)z_n + gn]_n.
\]

Suppose the driving system (1) starts at some \( x_0 \), system (13) starts at small \( \epsilon_0 \) and is subject to small noise. \( \epsilon_0 \) and the amplitudes of noise can be made sufficiently small to validate the linear approximation over a desirable time interval. As dictated by (13), \( z_n \) after \( k \) iterations is determined by

\[
z_k = \left\{ (1-g)^k \prod_{n=0}^{k-1} \frac{df(x)}{d\xi} \bigg|_{\xi=\epsilon^{(n)}(x)} \right\} \epsilon_0 + g \sum_{n=0}^{k-1} \left( (1-g)^{k-n-1} \prod_{m=n+1}^{k-1} \frac{df(x)}{d\xi} \bigg|_{\xi=\epsilon^{(m)}(x)} \right) \epsilon_n.
\]

Clearly the expressions in the braces are related to the local transversal multipliers \( \mu(x) \) defined as

\[
\mu(x) = (1-g)^l \prod_{k=1}^{l} \frac{df(x)}{d\xi} \bigg|_{\xi=\epsilon^{(k)}(x)}
\]

where \( x \) denotes the location on the synchronization manifold, \( l \) represents the duration of the trajectory. Equation (14) can be rewritten as

\[
z_k = \mu(x_0) \epsilon_0 + g \sum_{n=0}^{k-1} \mu_{k-n-1}(f^{(n)}(x_0)) \epsilon_n.
\]

Note that, by definition of \( \mu(x) \) and \( \lambda_\perp \), the two are related by \( \lambda_\perp = \lim_{n \to \infty} \frac{1}{l} \ln |\mu(x)| \). Thus, at large \( k \), the first term in (16) approaches zero exponentially, as long as \( \lambda_\perp < 0 \). Therefore, for the original coupled maps it means that if \( \lambda_\perp < 0 \) and there is no noise in the coupling channel, almost all initial deviations away from the synchronization manifold will decay and the systems will eventually synchronize. This observation is not new (see, for example, [18]), and it is in complete agreement with our numerical tests discussed above.

Unlike the first term, the second term contains contributions from local multipliers, even in the limit \( k \to \infty \). Suppose the noise has the characteristic amplitude \( \eta \ll 1 \). Then, apparently, it can be amplified to a finite scale \( \sim O(1) \) within the linear framework if some of the multipliers in the sum of expression (16) are of the order \( \sim 1/\eta \). We shall now show that in our system when \( g < 0.8 \) the multipliers \( \mu_{k-n-1}(f^{(n)}(x_0)) \) in (16) can be arbitrarily large for certain locations on the chaotic set of synchronized motions.
It follows from (5) that the region of the greatest transversal instability is the one where \( |d^f(x)/dx| \) is maximum. In our example, the derivative is maximum at the point \( x = 0 \) which is a fixed point of the driving map. The point \((x, y) = (0, 0)\) is a fixed point of the coupled system (1), (2). This point is embedded into the chaotic set of synchronized motions meaning that a chaotic trajectory in this set can come arbitrarily close to this point.

Now examine how the stability of the point \((x, y) = (0, 0)\) depends on \( g \). The point \( x = 0 \) in the driving system is unstable. Since the coupling is one-way, the point \((x, y) = (0, 0)\) will be unstable in the direction along the synchronization manifold, for all values of \( g \). Now how does its transversal stability change? The quite straightforward analysis shows that this fixed point is stable with respect to perturbations away from the synchronization manifold if \( g > g_0 = (\alpha - 1)/\alpha = 0.8 \). As \( g \) decreases, at \( g = g_0 \) the bifurcation occurs in which two new fixed points are born outside the synchronization manifold. Thus, at this moment the point \((x, y) = (0, 0)\) becomes transversally unstable and a trajectory appears that departs from the synchronization manifold exponentially rapidly.

Let us now come back to the mechanism of noise amplification in the linearized system (13). Suppose \( g < g_0 \) and thus the fixed point at the origin is transversally unstable. We can show that in this case, for any arbitrarily large \( G \) one can find \( r \) such that for every \( |x| < r \) there is such \( I \) that \( |\mu(x)| > G \) (see Appendix B for details). Thus, for points close to \((x, y) = (0, 0)\) the multipliers \( \mu(x) \) grow without bound. During its evolution on the chaotic set of synchronized motions the system can approach arbitrarily close to the point \((x, y) = (0, 0)\). Suppose there is noise of characteristic amplitude \( \eta \) in the channel. Then for arbitrarily small \( \eta \) the system will eventually pass so close to the point \((x, y) = (0, 0)\) that at some iteration \( |\mu(x)\eta_i| \sim 1 \). So arbitrarily small noise in the synchronization channel creates the opportunity for the system to deviate to a finite distance from the synchronization manifold \( x = y \). This scenario, in our example, explains why the coupled maps remain unsynchronized in the region \( g_0 < g < g_0 \).

The mechanism of noise amplification in the neighborhoods of periodic orbits is similar to the one we just considered. Thus, for robust synchronization of chaos it is crucial that all invariant sets embedded in the chaotic set of synchronized oscillations are transversally stable. This conclusion agrees with the results of our numerical simulation shown in Fig. 3 up to the following consideration. The robustness is determined by the behavior of a system in the presence of infinitesimal perturbations. Thus, the robustness issue should be most adequately illustrated by the curves corresponding to the smallest noise amplitudes. There are two reasons why these curves do not demonstrate a clear transition point at \( g = 0.8 \). One is that the outbursts of unsynchronized behavior close to this point are increasingly seldom and it becomes increasingly hard to collect adequate amount of data to properly estimate the maximum deviation. The second is that this transition is supercritical and the maximum deviation increases continuously from almost zero as the coupling strength decreases.

### III. Summary and Discussion

In this paper, we considered the influence of noise on synchronization in coupled chaotic maps. We showed that when fixed points or periodic orbits embedded in the chaotic set of synchronized motions are transversally unstable, infinitesimal noise can be amplified to a finite scale in the neighborhoods of such locations. This can lead to outbursts of unsynchronized behavior even when the coupling is such that the transversal or conditional Lyapunov exponent is negative. Some experimental work with nonlinear circuits [21], [22], [25] confirms that the transversal instability of fixed points in the neighborhood of chaotic sets of synchronized motions can lead to destabilization of synchronous operation in practice.

Thus, we established that having all negative global transversal Lyapunov exponents evaluated along the observed chaotic orbits is not sufficient to determine the conditions when two chaotic physical systems will display undisturbed synchronization. We showed that synchronization can only be robust when the whole chaotic set of synchronized motions, including all fixed points and periodic orbits embedded in it, is transversally stable. Thus, in practical studies of synchronized chaos, one cannot rely on the traditional transversal Lyapunov exponents analysis and should examine all potentially unstable invariant solutions within the chaotic set in the synchronization manifold. The mathematical foundations for this can be found in [23], and our exposition indicates how one is to carry out this augmented investigation in practice. The alternative approach to the analysis of transversal stability of the chaotic set of synchronized motions is based on the construction of Lyapunov functions (see for example [11], [26]–[28]).

In conclusion, we should note that, although in this paper we focused on the illustration of stability issues for the regime of synchronized chaos, the mechanism considered here is being studied in connection with such phenomena as “riddled” basins and “bubbling” of chaotic attractors [29]–[33]. In fact, the behavior that we considered is an example of “bubbling” behavior in a problem of chaos synchronization.

### Appendix A

#### Stability of Synchronized Chaos Based on Contraction Mappings

Using the ideas of the method of contraction mappings we shall show that for \( g > 0.8 \) the solution \( z = 0 \) in the system (1), (5) is stable for all invariant measures defined on the chaotic set of synchronized motions. This stability is guaranteed if there is a domain of initial conditions in the plane \((x, z)\) such that it encloses the chaotic set of synchronized motions and all trajectories in this domain are attracted by this chaotic set.

Let us consider the plane of variables \((x, z)\). On this plane we select a domain \( L_0 \) that contains the whole set of synchronized chaotic motions \( S_{SCM} \). After one iteration of the map, the domain \( L_0 \) will be transformed into \( L_1 \). If one can show that \( L_0 \supset L_1 \supset \cdots \supset L_n \supset S_{SCM} \), then this proves that \( S_{SCM} \) is an attractor for all trajectories starting in the domain \( L_0 \) [35]. Consider the domain \( L_0 \) in the form of a rectangle \( ABCD \) centered at the origin of the plane, see Fig. 6. Let the sides \( AB \) and \( CD \) be parallel to the \( x \)-axis and have length \( a \). The other sides \( BC \) and \( DA \) are parallel to the

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**Fig. 6.** Let the sides \( AB \) and \( CD \) be parallel to the \( x \)-axis and have length \( a \). The other sides \( BC \) and \( DA \) are parallel to the

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Fig. 6. The transformation of the rectangle ABCD (see Appendix A) after one iteration: (a) $g > g_0 = 0.8$; (b) $g < g_0$.

z-axis and have length $b$. We select the length $a$ such that $AB$ covers the chaotic attractor of the driving system (1), but less than $2\sqrt{\alpha + 1}/\alpha - \beta$. This guarantees that, as $n$ increases, the domain $L_n$ will shrink monotonically along the $z$-axis until it reaches in this projection the size the chaotic attractor in the driving system.

The transformation of the domain $L_0$ along the $z$ axis can be derived from the system (5). The edges of the domain are given by the lines $z = b/2$ and $z = -b/2$. In order to plot the images if these edges one has to multiply these values of $z$ by the coefficient $C(z) = (1 - g)[df(z)/dz]$, see (5). Therefore, if the absolute value of this coefficient is less than 1 within the domain $L_0$, then the image of $L_0$ will be entirely inside $L_0$ that is $L_0 \supset L_1$. Analysis of the function $f(x)$ shows that within the domain $L_0$ the maximum value of $|M|$ is reached at the point $x = 0$. The value of $M$ at this point is $M(0) = (1 - g)\alpha$. Thus, the stability of $S_{SCM}$ is guaranteed when $M(0) < 1$, i.e., $g > (\alpha - 1)/\alpha = 0.8$. Fig. 6(a) and (b) presents the sketches of the domains $L_0$ and $L_1$ for the cases $g > 0.8$ and $g < 0.8$, respectively.

Appendix B

Behavior of Characteristic Multipliers in the Neighborhood of the Transversally Unstable Fixed Point

In our discussion of the mechanism of noise amplification in our coupled maps, we noted that when the fixed point at the origin is transversally unstable, for any arbitrarily large $G$ one can find $r$ such that for every $x$ such that $|x| < r$ there is an $I$ such that $\mu(x) > G$. Let us now demonstrate that this is indeed so.

Introduce the point $x_0 > 0$ such that

$$
(1 - g) \frac{df(x)}{dx} \bigg|_{x = x_0} = 1.
$$

Such point can always be found if $g \leq (\alpha - 1)/\alpha$. Note that

$$
\frac{df(x)}{dx} \leq 0 \text{ for all } x \in [0, x^*_a].
$$

This means that for $x > 0$ the first derivative of $f(x)$ decreases monotonically. From this we can conclude that, as long as the system remains in the region $x < x_*$, its multipliers will grow with each iteration, since at each step $\mu_{n+1}(x) = \mu(x)A$ where $A > 1$.

Now, let us define two more numbers

$$
x' = f^{(n-1)}(x_*) \quad \text{and} \quad D = \frac{df^{(n)}(x)}{dx} \bigg|_{x = x^*}.
$$

Let us consider an arbitrary point $y$ such that $0 < y < x'$ and estimate its maximum multiplier. Let $n(r)$ be such that $f^{(n)(r)}(y) < x'$ and $f^{(n)(r)}(x') > x'$ so that the multiplier of the point grows until the $[n(r) + 1]^{th}$ iteration. By definition

$$
\mu_{n(r)}(r) = \prod_{k=1}^{n(r)} (1 - g) \frac{df^{(k)}(x)}{dx} \bigg|_{x = f^{(k)}(r)}.
$$

Thus

$$
f^{(k)}(x) \leq \alpha^k x.
$$

Let us define the integer $l(r)$ such that $\alpha^{l(r)} r < x'$ and $\alpha^{l(r)+1} r > x'$: $\log_{\alpha} (x'/r) - 1 < l(r) < \log_{\alpha} (x'/r)$. Then due to (22), $l(x) \leq n(r)$. In (20), all terms in the product are larger than 1, and therefore

$$
\mu_{n(r)}(r) \geq \prod_{k=1}^{l(r)} (1 - g) \frac{df^{(k)}(x)}{dx} \bigg|_{x = f^{(k)}(r)}.
$$

Further, due to (18) and since $f^{(k)}(r) < x'$, in (23) we have

$$
\frac{df^{(k)}(x)}{dx} \bigg|_{x = f^{(k)}(r)} \geq 1.
$$

Thus

$$
\mu_{n(r)}(r) \geq [(1 - g)D]^{l(r)} = M(r).
$$

Note that $M(r)$ is a monotonically decreasing function of $r$ because, by the definition of $D$, $(1 - g)D \geq 1$ and, by its definition, $l(r)$ is a monotonically decreasing function. Thus, for all $x < r$, $\mu_{n(x)}(x) > M(r)$.

By definition of $l(r)$, $l(r) \rightarrow \infty$ as $r \rightarrow 0$ and, thus, $M(r) \rightarrow \infty$ as $r \rightarrow 0$. Although we only considered $x > 0$, the results are transferred with almost no changes to the region $x < 0$, due to the symmetry of the problem. Therefore, by taking the inverse of function $M(r)$ we can find for any arbitrarily large number $G$ find $r$ so small that for all $|x| < r$ $\mu_{n(x)}(x) > M(r) = G$. 

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2Here and throughout the Appendix $f(\bullet)$ is the function defined by (3).
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REFERENCES


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