MUTUAL SYNCHRONIZATION OF CHAOTIC SELF-OSCILLATORS WITH DISSIPATIVE COUPLING

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In a case of two dissipative coupled electronic circuits with possible chaotic dynamics, results of experimental, analytical and computer studies are provided concerning bifurcations on the boundaries of the synchronization regime of chaotic self-oscillations.

1. Introduction

Oscillations of nonlinear dynamical systems possess a number of specific properties with very important applications in different fields of science and engineering [Nicolis & Prigogine, 1977, Velarde, 1982; Haken, 1983]. One of these properties is “synchronization”.

It is known that synchronization can occur not only with periodic oscillators but also with chaotic systems [Afraimovich et al., 1986; Fujisaka & Yamada, 1983]. For instance, in Gaponov-Grekhov et al. [1984] and Anichenko et al. [1986], it has been illustrated that in a chain of coupled oscillators along the half-line \([0, \infty)\), forced synchronization, i.e., one-way synchronization induced by the \(i\)th oscillator upon the \((i + 1)\)th one, leads to asymptotically stable values of KS entropy [Eckmann & Ruelle, 1985], dimension and power spectrum characteristics. Thus one may expect that synchronization due to mutual coupling, i.e., form \(i\)th to \((i + 1)\)th and vice versa, could lead to self-organization and homogeneous chaotic oscillations in a network of coupled cells with complex individual dynamics.

Recently, the problem of synchronous chaotic oscillations has attracted interest from different points of view given its potential applications [Pecora & Carroll, 1990 & 1991; Carroll & Pecora, 1991; Endo & Chua, 1991]. After a rather general definition of synchronization was given in Afraimovich et al. [1986], a number of analytical, experimental and numerical results about stability of synchronized chaotic oscillations and self-oscillations were obtained for coupled nonlinear oscillators with periodic forcing [Afraimovich et al., 1986; Verichev & Maksimov, 1989], Lorenz’s equations [Verichev, 1986], electronic circuits [Anichenko et al., 1991; Volkovskii & Rul’kov, 1989] and Selkov models [Badola et al., 1991]. However, possible bifurcation scenarios of transition to the synchronization regime demanded further study. For a first approach it suffices to use simple models to study these bifurcations. In this note we consider the typical evolution from nonsynchronized self-oscillations to synchronized ones with mutual coupling parameter in a case of two dissipative coupled electronic circuits exhibiting chaotic behavior.

We have experimentally studied the evolution of two coupled electronic circuits connected by means of a resistor \(R\) (Fig. 1). Let us start our considerations with a description of the main features of the dynamics of one of the circuits. The
dynamics of the circuit can be described by a three variable system of differential equations [Dmitriev et al., 1985; Volkovskii & Rul'kov, 1988]

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x - \delta y + z, \\
\dot{z} &= \gamma [F(x) - z] - \sigma y,
\end{align*}
\]

(1)

where \(x\) is the voltage at the capacitor \(C'\), \(y = J(L/C')^{1/2}\), \(z\) is the voltage at the capacitor \(C\), \(\tau = t/(LC')^{1/2}\), and

\[
\gamma = \frac{\sqrt{LC'}}{RC}, \quad \delta = r\sqrt{\frac{C'}{L}}, \quad \sigma = \frac{C'}{C},
\]

(2)

\[
F(x) = \begin{cases} 
0.528\alpha & \text{for } x < -1.2, \\
\alpha x(1-x^2) & \text{for } -1.2 < x < 1.2, \\
-0.528\alpha & \text{for } x > 1.2.
\end{cases}
\]

The model has the nice feature of being linear in the neighborhood of the origin; its growth is bounded when \(x\) exceeds a certain threshold and reinjects the signal at the upper level while maintaining symmetry. Yet it has symmetric and asymmetric attractors. Indeed for different values of the parameters, the circuit has a single symmetric one and asymmetric pairs (one or the other attractor in the pair is attained according to initial conditions). Projections of these strange attractors are given in Fig. 2. It has already been shown in Volkovskii & Rul'kov [1988], that chaotic properties of the self-oscillator are connected with bifurcations of pairs of saddle-focus Oo(0,0,0) homoclinic trajectories with positive saddle-focus value [Shilnikov, 1970]. Figure 3 provides the maximum nonzero Lyapunov exponent in the parameter range \(-12 < \alpha < 34\)

where we clearly see how it evolves from negative to positive values, thus illustrating the appearance of chaotic regions.

Although the behavior of our circuit is similar to Chua's circuit [Matsumoto et al., 1985 & 1988; Chua et al., 1986], note that our system has constant negative divergence of the phase flow, \(\text{div } u = -\gamma - \delta\). This enables us to check the results of our numerical calculation of the Lyapunov exponents. In Chua's case the divergence is stepwise constant with the actual value depending on the variable range explored. Both Chua's circuit and our circuit have Shilnikov homoclinic chaos.

2. Synchronization of Chaos in the Experiment

For the study of mutual synchronization we start by using the two coupled circuits with equal parameters. Then we set

Fig. 1. Block scheme of the two circuit (I & II) system. 1 denotes the nonlinear part. 2 & 3 are oscilloscopes. Component values are \(r_{1,2} = 0.407\, k\Omega\); \(C_{1,2} = 375\, n\Phi\); \(L_{1,2} = 233.7\, m\Phi\); \(R_{1,2} = 6\, k\Omega\); \(0 < R < 2.4\, k\Omega\); \(R_3 = 2.7\, k\Omega\); \(R_4 = 22\, k\Omega\); \(R_5 = 180\, k\Omega\); \(R_6 = 7.5\, k\Omega\); \(R_7 = 7.5\, k\Omega\). The diodes are of type 1N4148 and OP amps are of type LF356.

Fig. 2. \((x, z)\) projections of typical chaotic attractors produced by the circuit of a single self-oscillator. (a) Pair of nonsymmetric attractors, (b) symmetric attractor.

Fig. 3. Dependence of the maximum nonzero Lyapunov exponent on the parameter \(\alpha\), calculated for attractors of the system (1).
\[ \dot{x}_1 = x_2 + \frac{\epsilon}{2}(x_2 - x_1), \quad \dot{x}_2 = y_2 - \frac{\epsilon}{2}(x_2 - x_1), \]
\[ \dot{y}_1 = -x_1 - \delta y_1 + \gamma_1, \quad \dot{y}_2 = -x_2 - \delta y_2 + \gamma_2, \]
\[ \dot{z}_1 = \gamma[F(x_1) - z_1], \quad \dot{z}_2 = \gamma[F(x_2) - z_2] \]
\[ -\sigma y_1, \quad -\sigma y_2, \]  
where subscripts 1 and 2 correspond to different oscillators. The coupling parameter \( \epsilon \) is controlled by the resistor \( R \):
\[ \epsilon = 2 \frac{L}{R} \sqrt{C'}. \]

It is easy to see that a manifold
\[ x_1 = x_2, \]
\[ y_1 = y_2, \]
\[ z_1 = z_2 \]
exists in the six-dimensional phase space of (3). When the trajectory of the system lies on this manifold, the behavior of the two oscillators is the same as for each one alone. If the parameters of the self-oscillators are chosen in the region of chaotic behavior (see Fig. 3) and the manifold (5) is stable, then, according to the results of Afraimovich et al. [1986], the attractors on the manifold will correspond to the synchronization regime of chaotic self-oscillators. Then, in this regime, if we know how one self-oscillator behaves, then we know exactly the state of the other one in a given time. This, in our case, is the practical rule for synchronization, according to the general definition given in Afraimovich et al. [1986].

To study synchronization in the experiment, and just for illustration, the parameters of the circuits were chosen in a region where there is a symmetric strange attractor (Fig. 2b). Similar experimental findings have been obtained with an asymmetric attractor. The \((x_1, z_1)\) and \((x_1, x_2)\) projections of the attractors were studied using two oscilloscopes (denoted by 2 and 3 in Fig. 1). The synchronization regime corresponds to chaotic oscillations along the diagonal in the \((x_1, x_2)\) plane (Fig. 4a) which in the \((x_1, z_1)\) plane are depicted in Fig. 2b.

The only stable state found for \( \epsilon \) higher than some critical value \( \epsilon^* \) is the oscillatory regime in the manifold (5) (see Fig. 4a). It remains stable for \( \epsilon_{SM} < \epsilon < \epsilon^* \) but other stable states coexist with it in this region. For \( \epsilon < \epsilon_{SM} \) the oscillatory regime becomes unstable. Looking at the stable states outside the manifold, when \( \epsilon \) decreases and crosses the boundary \( \epsilon = \epsilon^* \), two stable fixed points, \( O^+ \) and \( O^- \), appear outside the manifold. Afterwards they lose stability undergoing an Andronov–Hopf bifurcation, and two periodic attractors, \( T^+ \) and \( T^- \), appear in the phase space. Projections of the attractors in the multistable regime are shown in Fig. 4b. Further decrease of \( \epsilon \) leads to a Hopf bifurcation of \( T^+ \) and \( T^- \), and afterwards there is a transition to chaos via quasiperiodicity (as the power spectrum shows). The bifurcation scenario described above shows that hysteresis exists on the boundary of the synchronization zone of chaotic self-oscillations. Thus to estimate the value of one of the boundaries of the chaotic synchronization regime \( \epsilon^* \) it suffices to investigate the stability and bifurcations of the fixed points \( O^+ \) and \( O^- \).

The analytical study of the system (3) shows that decreasing the coupling parameter, a bifurcation from the fixed point \( O_0 \) takes place at \( \epsilon_0 = \gamma(\alpha - 1)/(\sigma + \gamma \delta) \), and two other unstable fixed points, \( O^+ \) and \( O^- \), appear outside the manifold as illustrated in Fig. 5a. On the boundary \( \epsilon^* = 2\epsilon_0/3 \), two additional pairs of unstable fixed points arise as a result of bifurcation from \( O^+ \) and \( O^- \). Then \( O^+ \) and \( O^- \) become stable (Fig. 5b). These points are stable up to \( \epsilon = \epsilon_* = [\gamma(\alpha - 1) - (\gamma + \delta)(1 + \sigma + \delta \gamma)]/[3(\sigma + \gamma \delta)] \), where an Andronov–Hopf bifurcation takes place.

Fig. 4. \((x_1, x_2)\) projections of attractors taken from the oscilloscope 3 (see Fig. 1) when decreasing the coupling parameter. (a) Chaotic oscillations lying on the manifold, (b) attractors of multistable regime, (c) quasiperiodic attractors, (d) chaotic attractors.
LES\perp we introduce new variables
\[ \begin{align*}
X &= x_1 - x_2 \\
Y &= y_1 - y_2 \\
Z &= z_1 - z_2,
\end{align*} \]
which give us a linearized system of the form
\[ \begin{align*}
\dot{X} &= Y - \epsilon X, \\
\dot{Y} &= -X - \delta Y + Z, \\
\dot{Z} &= \gamma [F'(x_1(\tau))X - Z] - \sigma Y,
\end{align*} \]
where \( x_1(\tau) \) is a solution on a chosen attractor of the system (1). Calculation of LES for the system (7) gives us the LES\perp.

In the case of periodic self-oscillations and \( \epsilon = 0 \), only two maximum Lyapunov exponents of the full spectrum are zero. The introduction of the coupling makes one of them negative. This is connected with the bifurcation of a stable limit cycle on a torus. This behavior is in agreement with the usual synchronization of equivalent periodic self-oscillators where synchronization can be obtained for arbitrary small coupling.

In the case of chaotic self-oscillations the situation is different. For \( \epsilon = 0 \) we have two positive (\( \lambda_1^\parallel = \lambda_1^\perp > 0 \)), two zero and two negative Lyapunov exponents. As it can be seen in Fig. 6 when \( \epsilon \) grows, all components of LES\perp monotonically decrease. For \( \epsilon = \epsilon_{SM} \), \( \lambda_1^\perp \) crosses zero and for higher values of \( \epsilon \) becomes negative. Thus, in spite of local instability of the chaotic oscillations on the manifold, the coupling brings the synchronization of these oscillations, thus leading to local stability. The evolution of \( \lambda_1^\parallel \) and \( \lambda_1^\perp \) with \( \epsilon \), shows that the KS entropy of chaotic self-oscillations of the system (3) in the synchronization regime is given by \( \lambda_1^\parallel \) only. It becomes equal to the KS entropy for a single self-oscillator alone.

In numerical simulations we have seen (see Fig. 6) the rather smooth dependence of \( \epsilon_{SM} \) on the maximum Lyapunov exponent. Thus, if we consider the dependence of \( \epsilon_{SM} \) with parameters of the self-oscillator it will be rather complex and similar to Fig. 3 in the region of positive values.

To study the global behavior of the system (3) near the boundary \( \epsilon_{SM} \) we must consider, in the six-dimensional phase space, the dependence of the attractors on the coupling parameter. One of the results found is shown in Fig. 7 where the
dependence of the three maximum Lyapunov exponents on \( \epsilon \) can be seen. For small enough \( \epsilon \) there is a *hyperchaotic* regime (more than one positive Lyapunov exponents exist). As \( \epsilon \) is increased, bifurcations of the hyperchaotic attractors leads to transition to chaos, quasiperiodicity and afterwards to a stable periodic attractor. All these attractors are outside the manifold because the solutions on the manifold are unstable up to \( \epsilon = \epsilon_{SM} \). Then for values of \( \epsilon_{SM} < \epsilon < \epsilon' \), at least two different attractors coexist in phase space, one of them is periodic and the other is the chaotic attractor lying on the manifold that corresponds to the synchronization regime. For higher values of \( \epsilon \) the synchronization regime remains locally stable and the dependence of the Lyapunov exponents on \( \epsilon \) depicted in Fig. 7d is similar to that shown in Fig. 7b.

### 4. Synchronization of Nonequivalent Chaotic Self-Oscillators

Generally, in practical uses, there is no possibility of performing an experiment with perfectly identical self-oscillators. Thus, it is interesting to explore how the behavior of the system (3) changes when small differences between the self-oscillators exist.

The coupled circuits with different parameters can be written in the form

\[
\begin{align*}
\dot{x}_1 &= y_1 + \frac{\epsilon}{2}(x_2 - x_1), \\
\dot{y}_1 &= -x_1 - \delta_1 y_1 + z_1, \\
\dot{z}_1 &= \gamma_1[F_1(x_1) - z_1] - \sigma_1 y_1, \\
\dot{x}_2 &= y_2 - \frac{\epsilon}{2}(x_2 - x_1), \\
\dot{y}_2 &= -x_2 - \delta_1 y_2 + z_2, \\
\dot{z}_2 &= \gamma_2[F_2(x_2) - z_2] - \sigma_2 y_2,
\end{align*}
\]

where, for simplicity, \( L \) and \( C' \) are assumed to be the same as before.

Now the model does not have, in general, the invariant manifold (5), but the experiment shows that the synchronized oscillations change little for sufficiently small differences between the parameters of the circuits. In order to estimate the changes, let us consider the following new parameters:

\[
\begin{align*}
\delta &= (\delta_1 + \delta_2)/2, \\
D_\delta &= (\delta_1 - \delta_2)/2, \\
\gamma &= (\gamma_1 + \gamma_2)/2, \\
D_\gamma &= (\gamma_1 - \gamma_2)/2, \\
\sigma &= (\sigma_1 + \sigma_2)/2, \\
D_\sigma &= (\sigma_1 - \sigma_2)/2, \\
\alpha &= (\alpha_1 + \alpha_2)/2, \\
D_\alpha &= (\alpha_1 - \alpha_2)/2,
\end{align*}
\]
and the variables (6). The perturbed system is now

\[
\dot{X} = Y - \epsilon X, \\
\dot{Y} = -X - \delta Y + Z - \Delta_y, \\
\dot{Z} = R [f(x_1) - f(x_2)] - \gamma Z - \sigma Y + \Delta_z,
\]

where

\[
\Delta_y = D_\delta (y_1 + y_2), \quad R = \alpha \gamma + D_\alpha D_\gamma, \\
\Delta_z = (\gamma D_\alpha + \alpha D_\gamma)(f(x_1) + f(x_2)) \\
- D_\gamma (z_1 + z_2) - D_\sigma (y_1 - y_2), \\
f(x_i) = F_i(x_i)/\alpha_i, \quad i = 1, 2.
\]

\(x_i, y_i\) and \(z_i\) are the solutions of the system (8). Experiments and numerical simulations of the system (8) show that the solutions remain bounded. This means that if perturbations of parameters are small enough then \(\Delta_y\) and \(\Delta_z\) are small too.

Let us consider a positive definite function

\[V = \frac{1}{2} (X^2 + Y^2 + \frac{Z^2}{\sigma}).\]

Its time derivative for the system (10) is

\[
\frac{dV}{d\tau} = -\epsilon \left( X - \frac{ZR\phi(x_1, x_2)}{2\sigma\epsilon} \right)^2 - \delta \left( Y - \frac{\Delta_y}{2\delta} \right)^2 \\
- P(\epsilon) \left( Z - \frac{\Delta_z}{2P(\epsilon)} \right)^2 + \frac{\Delta_y^2}{4\delta} + \frac{\Delta_z^2}{4P^2(\epsilon)},
\]

where

\[
\phi(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}, \quad |\phi(x_1, x_2)| < \phi_0 = 3.4, \\
P(\epsilon) = \frac{\gamma}{\sigma} - \frac{R^2 \rho^2}{4\sigma^2 \epsilon}.
\]

If \(\Delta_y\) and \(\Delta_z\) are zero, then, for \(P(\epsilon) > 0\), the function (12) will be a Lyapunov function and the fixed point \((0, 0, 0)\) of the system (10), as well as the manifold (5), will be stable for all initial conditions. This situation occurs in model (3). This means that, for equivalent circuits, the synchronization of chaotic oscillations regimes will be guaranteed for \(\epsilon > \gamma \alpha^2 \phi_0^2/4\sigma\). Note that, at least for the case of coupled identical self-oscillators, the Lyapunov function is the excess dissipation needed to take the system away from the steady state [Ross et al., submitted].

If \(\Delta_y\) or \(\Delta_z\) are not zero, then from Eq. (13) it follows that, for \(P(\epsilon) > 0\), all the trajectories of the system (10) go into an ellipsoid centred in \((0, 0, 0)\). The length of the axes of the ellipsoid are proportional to \(D_\alpha, D_\gamma, D_\delta\) and \(D_\sigma\). Thus, for small enough parameter perturbations, the attractor that corresponds with synchronized oscillations appears in the neighborhood of the purely homogeneous oscillations.

Numerical simulations show that the bifurcation scenario of the transition to synchronization regime does not change much if the perturbations of the parameters lie in a region with analogous attractors. Comparison of Figs. 7d and 7c shows that the dependence of LES on the coupling parameter remains qualitatively the same before and after such perturbations.

5. Conclusion and Outlook

It has been demonstrated, by using dissipative coupled nonlinear circuits with individual chaotic behaviors, that the local and global stability boundaries of the synchronization regime of chaos can be located at different coupling parameter values. This is connected with the bifurcations of some attractors in the global regions of the phase space.

To clearly see the nature of the synchronization regime of chaos, bifurcations on the boundaries demand a study with, at least, a two-parameter space. Then all difficulties are due to the complex structure of the bifurcations of just a single chaotic oscillator taken alone. Indeed, in this case a small variation of a parameter of the oscillator can change completely the state of the system.

Note that the introduction of a coupling between chaotic self-oscillators can lead not only to more complex behavior or to the stabilization of the chaotic self-oscillations, but also to the appearance of simple types of attractors like, for example, limit cycles as illustrated in Fig. 8. This result bears similarity with the case of coupled cell cultures of *Dictyostelium* amoebae suspensions [Halloy et al., 1990]. The experiments showed, somewhat unexpectedly, the occurrence of rather regular oscillations when different possible chaotic cells, where coupled, provided a fraction of cells that behave in a periodic manner. The cells shared a common intermediate chemical which is like the resistor in our case.

Finally, let us mention that earlier authors have reached similar conclusions for nonlinear systems with appropriate external controls, e.g., with feedback [Singer et al., 1991] or with harmonic forcing.
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Fig. 8. (a) and (b) Seemingly chaotic attractors of uncoupled circuits “1” and “2”, (c) The results of coupling is a stable limit cycle.

[Braiman & Goldhirsch, 1991]. Indeed, as already discussed by several authors [Ott et al., 1990], in an experiment using a controller, one can obtain an actually stable nonlinear oscillation in a possible chaotic regime of the system, or vice versa.

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