ROBUSTNESS OF SYNCHRONIZED CHAOTIC OSCILLATIONS

NIKOLAI F. RULKOV
Institute for Nonlinear Science,
University of California at San Diego, La Jolla, CA 92093, USA

MIKHAIL M. SUSHCHIK
Department of Physics and Institute for Nonlinear Science,
University of California at San Diego, La Jolla, CA 92093, USA

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In this paper we address the issue of robustness of synchronized chaotic oscillations in coupled systems. One frequently observes in physical experiments that synchronized chaotic oscillations are occasionally interrupted by brief incidents of unsynchronized behavior. By numerical simulations we show that, under certain circumstances, the regime of synchronized chaos is very sensitive to even small noise and to slightest differences between parameters of coupled systems. As a result of such sensitivity, these small perturbations lead to non-steady, “bursting” synchronization. Using experiments with nonlinear electronic circuits and analytical and numeric analyzes of their ODE model, we study certain bifurcations associated with fixed points and limit cycles in the synchronized chaotic attractor. We establish the connection between these bifurcations and the appearance of the outbursts of unsynchronized behavior. We illustrate a mechanism of formation of complicated invariant sets of trajectories that are associated with the dynamics during such outbursts.

1. Introduction

One of the most important manifestations of synchronized chaos is the regime in which two coupled chaotic systems exhibit identical chaotic oscillations. Let x and y be vectors in the state spaces X and Y of two coupled systems. The regime of identical chaotic oscillations in these coupled systems is associated with the existence of the invariant manifold of synchronized motions x = y and of the invariant chaotic set located in this manifold. This chaotic set is formed by the trajectories of the chaotic attractor of the dynamical system that is defined in the manifold x = y. This manifold (synchronization manifold) forms a hyperplane in the phase space \( U = X \oplus Y \) of the combined system. When the invariant chaotic set in synchronization manifold is an attractor in the phase space \( U \), the systems can produce sustained synchronized chaotic oscillations. Since the first observations [Pujisaka & Yamada, 1983; Afraimovich et al., 1986; Pecora & Carroll, 1990] of the synchronized chaos, the regime of identical chaotic oscillations received lots of attention due to its possible applications for secure communications [Pecora & Carroll, 1990; Kocarev et al., 1992; Cuomo & Oppenheim, 1993; Volkovskii & Rulkov, 1993; Halle et al., 1993; Kocarev & Parlitz, 1995], to control of systems with chaotic dynamics [Ott et al., 1990; Pyragas, 1992; Rulkov et al., 1994; Wu & Chua, 1994] and as a possible mechanism of transition to a low-dimensional chaos in systems with many degrees of freedom [Gaponov-Grekho et al., 1984; Winful & Rahman, 1990].

The central question in the theory of synchronized chaotic oscillations is “When does the
invariant chaotic set in the synchronization manifold become an attractor in the phase space of coupled systems?". The condition that this invariant chaotic set is an attractor implies that its trajectories are stable with respect to small perturbations transversal to the synchronization manifold. This transversal stability of the invariant chaotic set of synchronized motions has been considered as a crucial issue since the first works on synchronization of chaos [Fujsaka & Yamada, 1983; Afraimovich et al., 1986; Pecora & Carroll, 1990; Pikovsky & Grassberger, 1991]. Today the mathematical foundation of the linear transversal stability theory has been clearly established and spelled out in the recent paper by Ashwin et al. [1996]. However one frequently encounters serious difficulties when he attempts to apply mathematical findings in practice. Two practical criteria most frequently used for the analysis of transversal stability of synchronized chaotic motions are the Lyapunov function criterion and the criterion based on the analysis of transversal (conditional) Lyapunov exponents. The criterion based on the construction of Lyapunov functions in some cases allows one to prove that all trajectories in the phase space of coupled systems are attracted by the manifold of synchronized motions (see, for example, [Verichev, 1986; Rulkov et al., 1992; Wu & Chua, 1994; Rodrigues, 1994]). Although the existence of such Lyapunov function guarantees the onset of synchronization, it is not a regular approach since there is no general procedure for the construction of the Lyapunov function for an arbitrary system.

The use of transversal Lyapunov exponents for the analysis of chaos synchronization was proposed in [Fujsaka & Yamada, 1983; Pecora & Carroll, 1990]. In contrast with the Lyapunov functions, the analysis of transversal Lyapunov exponents, as described in [Fujsaka & Yamada, 1983], is quite straightforward and can be easily employed even for rather complicated systems. In conventional methods of numerical evaluation of Lyapunov exponents of an invariant chaotic set, these exponents are computed along a sufficiently long typical chaotic trajectory in this set. For the last few years the transversal Lyapunov exponents had become the most frequently used tool for analysis of the transversal stability of synchronized chaotic motions. However it has been pointed out [Pikovsky & Grassberger, 1991; Ashwin et al., 1994; Ott & Sommerer, 1994; Brown et al., 1994; Heagy et al., 1995; Gauthier & Bienfang, 1996] that in practice, the negativeness of numerically computed Lyapunov exponents does not always guarantee the onset of synchronized motions.2

It will be shown in this paper that, even in the cases where all transversal Lyapunov exponents (computed using conventional methods) are negative, there may exist atypical trajectories in the immediate vicinity of the manifold of synchronized motions which depart from this manifold exponentially fast. The appearance of such trajectories is responsible for the transversal destabilization of synchronized chaotic motions. Synchronized chaotic motions become unstable in the sense that vanishing noise in the coupling link and/or internal noise in the coupled systems and/or small mismatch between parameters of the systems lead to the loss of synchronization and to the bubbling behavior discussed in [Ashwin et al., 1994; Brown et al., 1994; Heagy et al., 1995]. We shall say that the regime of synchronized chaotic oscillations is robust when small perturbations lead to small deviations from the synchronization at all times. When infinitesimally small perturbations lead to finite scale outbursts of desynchronized behavior, we shall say that the regime of synchronized chaotic oscillations is not robust.

It was shown in a few recent papers [Ashwin et al., 1994; Heagy et al., 1995; Hunt & Ott, 1996; Gauthier & Bienfang, 1996] that one mechanism of transversal destabilization of synchronized motions is connected with the appearance of transversally unstable invariant limiting sets (in particular, periodic orbits and/or fixed points) embedded in the invariant chaotic set of synchronized motions. However the transition from robust synchronization to the unsteady synchronization with outbursts facilitated by such limiting sets was not studied. In this paper we focus on bifurcations scenarios which lead to the loss of robustness of the synchronized chaotic oscillations. We consider an example where these bifurcations are connected with saddle fixed points and limit cycles adjoining the chaotic

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1We shall sometimes call these exponents the global transversal Lyapunov exponents.

2Note that rigorous criteria of transversal stability based on Lyapunov exponents can be developed [Ashwin et al., 1996]. However, such criteria involve consideration of all invariant measures on the invariant chaotic set of synchronized motions. In practice this analysis may be very difficult, if not impossible, to carry out.
trajectories in the synchronization manifold. We follow the evolution of the characteristic motions that are associated with the desynchronization outbursts and present a few bifurcation scenarios of how these motions can increase their complexity. We demonstrate that the existence of transversally unstable limiting sets affects some quantitative characteristics of the systems behavior such as the distribution of local transversal Lyapunov exponents and the distribution of "escape points" on the invariant chaotic set of synchronized motions.

In our analysis we use directionally coupled electronic circuits shown in Fig. 1 (see Appendix A for details). We shall study this system analytically and numerically using its ODE model. In parallel, we shall present the results of experiment which helps us to establish the physical relevance of our theoretical analysis.

The dynamics of the coupled circuits shown in Fig. 1 is very well modeled by the following equations:

**driving system:**

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - \delta x_2 + x_3 \\
\dot{x}_3 &= \gamma(x_1 - x_3) - \sigma x_2
\end{align*}
\]  

(1)

**response system:**

\[
\begin{align*}
\dot{y}_1 &= y_2 - g(y_1 - x_1) \\
\dot{y}_2 &= -y_1 - \delta y_2 + y_3 \\
\dot{y}_3 &= \gamma(y_1 - y_3) - \sigma y_2
\end{align*}
\]  

(2)

where \(\alpha, \gamma, \delta\) and \(\sigma\) characterize the individual dynamics of the circuits. Note that Eqs. (1) and (2) are invariant with respect to transformation \((x, y) \rightarrow (-x, -y)\). The coupling between the systems is characterized by the parameter \(g = \frac{1}{R_2} \sqrt{\frac{L_2}{C_2}}\).

In the experiment the coupling strength is controlled by the resistor \(R_c\), see Fig. 1. The connection between the parameters of the models and real physical parameters of the circuits, as well as the values of parameters used throughout the paper, can be found in Appendix A.

It is easy to see that there exist the three-dimensional invariant manifold

\[
\begin{align*}
x_1 &= y_1 \\
x_2 &= y_2 \\
x_3 &= y_3
\end{align*}
\]  

(3)

in the six-dimensional phase space of the coupled systems (1) and (2). In certain range of parameters \(\alpha, \gamma, \delta\) and \(\sigma\) this manifold contains the invariant chaotic set that is an exact copy of the chaotic attractor in the driving system (1). This invariant chaotic set is the image of identical chaotic oscillations in the circuits. When the coupling parameter \(g\) is larger than some critical value, this set is transversally stable and the systems exhibit robust synchronized chaotic oscillations. Experimentally this regime is observed as the equality between the voltage waveforms measured at similar points in the circuits. At slightly lower values of the coupling \(g\), we experimentally observed that the circuits stay synchronized for very long times but occasionally

\[3\] Local transversal Lyapunov exponents are introduced following the definition of global transversal Lyapunov exponents given by Fujisaka and Yamada except that they are evaluated along a finite segment of a chaotic trajectory.
experience short-term outbursts of unsynchronized behavior.

To reveal the nature of this bursting behavior we consider systems (1) and (2) with parameter $\alpha \approx 25.361$ (see Appendix A for the values of other parameters). The $(x_1, x_2)$ projection of the attractor generated by the system (1) is shown in Fig. 2(a). We choose the coupling to be $g = 1.1$. The largest transversal Lyapunov exponent computed numerically at this value of the coupling parameter is $\approx -0.03$ while the largest Lyapunov exponent of the attractor of the driving system was $\approx 0.1$. The linearized systems and the procedure used for the computation of the Lyapunov exponents can be found in Appendix B. When the systems (1) and (2) are integrated numerically without added noise and with perfectly matched parameters, the plot of $y_1(t)$ versus $x_1(t)$ following the transient is a sharp diagonal $y_1(t) = x_1(t)$.

As one can see from Eqs. (1) and (2), the observation of this diagonal indicates that the systems evolve in the synchronization manifold (3). At the same time in the physical experiment we observe a very different behavior. We conducted the experiment with the same coupling strength and the same values of $\gamma$, $\delta$ and $\sigma$ as in numerical simulations. The parameter $\alpha$ in the experiment was selected to be equal to 25.8. Both the model with $\alpha = 25.361$ and the experimental system with $\alpha = 25.8$ are characterized by being close to the bifurcation of appearance of similarly shaped simple homoclinic orbits. This is illustrated in Figs. 2(b) and 2(d). Moreover, for these values of parameters of the model and of the experimental setup, the shape of the experimentally observed attractor is very close to that of the attractor obtained by numerical integration [compare Figs. 2(a) and 2(c)]. Although in the numerical simulations we observe
perfect synchronization at $g = 1.1$, in the physical experiment conducted with the same value of the coupling parameter there are outbursts of unsynchronized behavior, see Fig. 3(c).

We found that this bursting behavior can be reproduced numerically with the same parameters as in the previous numerical simulations, if a noise of very small amplitude is added to the variables in the integration process. The integration of the Eqs. (1) and (2) with the addition of small noise models the behavior of the coupled circuits in a realistic setting since in a physical experiment sources of small noise are always present. The same bursting behavior was observed when we integrated the equations without noise but with a very small mismatch of the parameters. Figure 3(a) shows the bursting behavior obtained by numerical integration of Eqs. (1) and (2) with parameters $\alpha$ in the coupled systems different by 0.4%. It is important that the dynamics of the systems during the outbursts does not depend on the character of perturbations. The figure obtained with noise added in the integration process and the one produced with the combination of noise and parameter mismatch can be nearly indistinguishable from Fig. 3(a). As the amplitude of the perturbations becomes smaller, the outbursts away from the synchronization manifold become less and less frequent. However, our numerical simulations show that, no matter how small this perturbation is, the trajectory will depart from the synchronization manifold to a finite distance, provided the system is observed for a sufficiently long time. Therefore, in this case the synchronization of chaos is not robust.
The analysis of locations on the attractor where disruptions of synchronization occur shows that there are “disruptive” regions on the attractor where the trajectory usually goes away from the manifold. There are also “quiescent” regions where the trajectory almost never escapes from the synchronization manifold. We show that, in our example, the “disruptive” regions are characterized by large positive values of local transversal Lyapunov exponents and that such regions are related to transversally unstable limiting sets embedded in the invariant chaotic set in the synchronization manifold. In Sec. 2 we consider a scenario in which bifurcations associated with a fixed point embedded in this chaotic set lead to the loss of the robustness. This occurs through the appearance of trajectories which escape from the manifold. We investigate characteristic trajectories that outline the structure of trajectories segments during the desynchronization outbursts. A possible mechanism of formation of complicated sets outside the manifold is discussed. In Sec. 3 we briefly describe a similar scenario of loss of robustness which involves bifurcations of saddle limit cycles. We discuss an experimentally observed mechanism of formation of complicated sets outside the manifold through the sequences of period doubling bifurcations of saddle limit cycles.

2. Loss of Robustness Facilitated by Transversally Unstable Fixed Points

The reason why infinitesimally small perturbations can lead to destabilization of synchronized chaotic oscillations can be most clearly illustrated when there is a saddle fixed point embedded in the chaotic attractor of the driving system. Although this is a rather specific situation, in this example it is much easier to follow the bifurcations that lead to the loss of robustness of synchronized chaotic motions.

To study the scenario of how the regime of synchronized chaotic oscillations ceases to be robust, we analyze certain bifurcations in the models (1) and (2) with \( \alpha \approx 25.361 \) (the values of \( \delta, \gamma, \sigma \) are given in Appendix A). We start with the description of fixed points and homoclinic orbits in the driving system.\(^4\)

Then we turn to the analysis of the coupled systems and discuss the bifurcation that leads to the appearance of transversally unstable fixed point embedded in the invariant chaotic set of synchronized motions. We observe the birth of new fixed points that move away from the synchronization manifold as the coupling parameter decreases. We show that, in our example, there exist a family of homoclinic and heteroclinic orbits associated with these fixed points. We discuss a mechanism of formation of complicated sets of trajectories in the vicinity of these orbits and show the connection between these trajectories and the dynamics during the outbursts of desynchronization.

2.1. Fixed points and homoclinic orbits in the driving system

The driving system (1) has three fixed points \( O_0(X), O_R(X) \) and \( O_L(X) \) which are saddle-foci. The fixed point \( O_0(X) \) is located at the origin of the phase space and has one-dimensional unstable manifold \( W^u_1[O_0(X)] \) and two-dimensional stable manifold \( W^s_2[O_0(X)] \). This fixed point will play the central role in our discussion. At the values of parameters that we have chosen, a trajectory on the chaotic attractor can pass infinitely close to \( O_0(X) \). The shape of the chaotic attractor obtained by numerical integration is shown in Fig. 2(a). Furthermore, there exist two simple homoclinic trajectories, \( H^L_0(X) \) and \( H^R_0(X) \), in the phase space of the driving circuit that originate and end at the saddle-focus \( O_0(X) \) [see Fig. 2(b)]. Similar attractor and homoclinic orbits were observed experimentally. These experimental results are shown in Figs. 2(c) and 2(d). To visualize the homoclinic orbits in the experiment we repeatedly suppress the oscillations in the driving circuit so that its consequent evolution starts at the fixed point \( O_0(X) \). Then we examine the shape of the trajectory that starts at the saddle-focus \( O_0(X) \) and moves along its one-dimensional unstable manifold. When the parameters are such that the system has the homoclinic orbit, this trajectory returns to the saddle-focus \( O_0(X) \) along the two-dimensional stable manifold of this fixed point. When the parameters are slightly detuned, and the homoclinic orbit is destroyed, then this trajectory either undershoots or overshoots the two-dimensional stable manifold of \( O_0(X) \). This can be easily seen in the experiment. The details of this experimental technique are discussed in [Rulkov & Volkovskii, 1995; Wu & Rulkov, 

\(^{4}\text{We selected parameters of the driving system so that it has two simple homoclinic orbits, because their existence facilitates a more detailed analysis of characteristic motions in the coupled systems.} \)
It should be understood that in an experiment it is impossible to achieve the perfect symmetry of the studied system. As the result of this experimental imperfection, the parameters at which the homoclinic orbit $H_L^c(X)$ exists are slightly different from the setting at which we experimentally observed the homoclinic orbit $H_R^c(X)$. In the experiment we tuned an auxiliary parameter of the nonlinear converter (the resistor Rn3, see Appendix A) so that both homoclinic orbits are observed at the same value of $\alpha$. Figure 2(d) shows the result of the experiment with $\alpha = 25.8$. Both left and right trajectories start at the origin and evolve along the one-dimensional unstable manifold of the fixed point $O_0(X)$. Although the left trajectory slightly undershoots the stable manifold of the fixed point $O_0(X)$, both left and right trajectories visualize very well the shape of the corresponding homoclinic orbits ($H_L^c(X)$ and $H_R^c(X)$).

It can be shown that if the parameters of the driving system (1) satisfy the condition

$$\alpha > \alpha_\Sigma = 1 + (\delta + \gamma)[2(\delta + \gamma)^2 + 1 + \sigma + \delta \gamma]/\gamma, \quad (4)$$

then the saddle-focus $O_0(X)$ has a positive saddle-focus value, $\Sigma_{sf} = \lambda_1 + \text{Re}\lambda_{2,3} > 0$ where $\lambda_{2,3}$ are two complex-conjugate eigenvalues of the fixed point and $\lambda_1 > 0$ is the real one. With chosen parameters values we have $\alpha_\Sigma \approx 14.56$, and thus the condition (4) is satisfied. Therefore, the set of trajectories which exists in the vicinity of the homoclinic orbits $H_L^c(X)$ and $H_R^c(X)$ possesses the properties which are guaranteed by Shil'nikov's theorem [Shil'nikov, 1970]. In our case it is important that: (i) this set includes a countable set of unstable periodic orbits which may be found in any close vicinity of the homoclinic orbits and (ii) under small variations of the parameters of the system, these parameters pass through a family of bifurcation values at which the system has subsidiary homoclinic orbits [Belyakov, 1984; Glendinning & Sparrow, 1984] associated with the fixed point $O_0(X)$. This implies that the bifurcations corresponding to the existence of homoclinic orbits in our system are typical when its chaotic attractors contain the saddle fixed point $O_0(X)$. Therefore our particular choice of the parameters values is not unique and reflects important for us dynamical properties for a whole domain of parameters, including the parameters values at which the simple homoclinic orbits $H_L^c(X)$ and $H_R^c(X)$ do not exist. The role of these homoclinic orbits will be reviewed at the end of this section.

### 2.2. Bifurcations of fixed points in the full phase space

Now we turn to the analysis of bifurcations in the six-dimensional phase space $U$ of the combined driving-response system with identical parameters. Clearly, in this six-dimensional phase space there are three fixed points $O_0(U)$, $O_R(U)$ and $O_L(U)$ located in the synchronization manifold (3), see Fig. 4.

Since the fixed point $O_0(U)$ is embedded in the invariant chaotic set of synchronized motions, we expect that bifurcations associated with this point may have a serious impact on the transversal stability of this invariant set. When the coupling is strong ($g > g_c = \gamma(\alpha-1)/(\sigma+\gamma\delta) \approx 3$), the unstable manifold $W^u_1(O_0(U))$ of the fixed point $O_0(U)$ is one-dimensional and is directed along the synchronization manifold (3), see Fig. 4(a). Thus this fixed point is stable with respect to perturbations transversal to the manifold of synchronized motions. Our numerical simulations and experimental studies show that in this case, when some small noise is present or the parameters of either driving or response circuit are slightly detuned, the trajectory stays close to the unperturbed manifold of synchronized motions at all times. The maximum distance to which the trajectory departs from the unperturbed synchronization manifold increases continuously from zero as the perturbation amplitude is increased. In this case the regime of synchronized chaotic oscillations is robust.

At $g = g_c$ the fixed point $O_0(U)$ undergoes a bifurcation and the unstable manifold of this point becomes two-dimensional due to the appearance of an additional unstable direction transversal to the synchronization manifold. As a result of this bifurcation, two new fixed points, $O_+(U)$ and $O_-(U)$, are born at the origin, see Fig. 4(b). As the coupling decreases, these new fixed points move away from the synchronization manifold. Note that the projections of all three fixed points $O_0(U)$, $O_+(U)$ and $O_-(U)$ onto the phase space, $X$, of the drive system correspond to the saddle-focus $O_0(X)$. By analysis of the eigenvalues of the fixed points $O_+(U)$ and $O_-(U)$ we found that there is an interval of the coupling parameter values $g^*_H < g < g_c$ where these fixed points have one-dimensional unstable manifolds $W^u_{1}[O_-(U)]$ and $W^u_{1}[O_+(U)]$. These manifolds...
are due to the existence of the unstable manifold of the fixed point \( O_0(X) \) in the driving system (1). \( g^*_H \) is the value of the coupling at which the off-manifold fixed points \( O_+(U) \) and \( O_-(U) \) suffer the Hopf bifurcation. Using Routh–Hurwitz criteria, we found that for chosen parameters \( g^*_H = 0.972 \).

### 2.3. Homoclinic and heteroclinic trajectories in the full system

We shall now argue that for \( g^*_H < g < g_c \) the existence of homoclinic orbits in the phase space \( X \) of the driving system results in the existence of homoclinic and heteroclinic trajectories outside the synchronization manifold in the phase space \( U \).

Let \( O_0(Y), O_+(Y) \) and \( O_-(Y) \) be the projections of the fixed points \( O_0(U), O_+(U) \) and \( O_-(U) \) onto the state space \( Y \) of the response system. Let us now consider the response system (2) driven by \( x_1(t) = 0 \). It can be shown analytically that in this case \( O_+(Y) \) and \( O_-(Y) \) are stable states of the response system for \( g_H^* < g < g_c \). Moreover, \( O_+(Y) \) and \( O_-(Y) \) are the only attractors which we found in our physical experiments and in numerical simulations with different initial conditions. Therefore, for our purposes we shall assume that these states are globally stable when \( x_1(t) = 0 \).

Let us now turn back to the evolution of the coupled systems. Suppose the coupled systems are started close to \( O_-(U) \) on the one-dimensional unstable manifold \( W^u(O_-(U)) \). This implies that the driving system moves along one of its homoclinic orbits \( \{H_0^R(X)\text{ or }H_0^L(X)\} \). Therefore, at \( t \to +\infty \) the driving system returns to the point \( O_0(X) \). Due to the global stability of \( O_+(Y) \text{ or } O_-(Y) \) with \( x_1(t) = 0 \) the final state of the response system must be either of these two states. Thus the only two possible nontrivial evolution trajectories starting at \( O_-(U) \) are the homoclinic orbit \( H_-(U) \) and the heteroclinic trajectory \( H^+_-(U) \) which connects \( O_-(U) \text{ and } O_+(U) \). Both these trajectories were observed in numerical simulations, see Fig. 3(b), and physical experiment, see Fig. 3(d). The homoclinic orbit \( H^-_-(U) \) is observed when the driving system is started along \( H_0^R(X) \) and the heteroclinic trajectory \( H^+_-(U) \) is observed when the driving system moves along \( H_0^L(X) \). It follows from the symmetry of the systems (1) and (2) that there are also homoclinic orbits \( H^+_+(U) \) associated with the fixed point \( O_+(U) \) and the heteroclinic trajectory \( H^+_+(U) \).

Similarly, it can be shown that there exist heteroclinic trajectories, \( H^+_0(U) \text{ and } H^-_0(U) \), that connect the fixed point \( O_0(U) \) at the origin with the off-manifold fixed points \( O_+(U) \text{ and } O_-(U) \). All homoclinic and heteroclinic trajectories considered in this section can be thought of as the skeleton for the set of trajectories associated with the bursting behavior. The relevance of these trajectories to the analysis of the off-manifold dynamics of the system during the outbursts of desynchronization is supported by the Fig. 3 [compare Figs. 3(a) and 3(c) with Figs. 3(b) and 3(d)]. One can see that there are many segments of trajectories during the outbursts which pass close to homoclinic and heteroclinic trajectories discussed in this section.5

5 Naturally, the conclusion that the outbursts segments pass close to the homoclinic and heteroclinic trajectories cannot be made by analysis of a single projection of these trajectories. We considered other projections as well and verified our conclusion.
2.4. Formation of complicated sets outside the synchronization manifold

To characterize the off-manifold dynamics associated with the outbursts, we intend to apply Shil’nikov’s theory to the homoclinic trajectories $H_-(U)$ and $H_+(U)$. To apply this theory, we classify the off-manifold fixed points by their least stable invariant manifolds in six-dimensional phase space. For the values of $g$ from the interval $g_{n-f} < g < g_c$ ($g_{n-f} \approx 2.2$, $g_c \approx 3$), the $O_-(U)$ and $O_+(U)$ are saddle-nodes with one-dimensional unstable manifolds. At $g = g_{n-f} \approx 2.2$ these points become saddle-foci, still with one-dimensional unstable manifolds. Analysis of the eigenvalues associated with the least stable manifolds of these saddle-foci shows that their saddle-focus values are positive.

The theoretical investigation of trajectories sets existing in the vicinity of homoclinic orbits in high-dimensional ($D > 3$) phase space was done in [Ovsyannikov & Shil’nikov, 1987]. The theory developed therein is a generalization of the well-known Shil’nikov’s theorem [Shil’nikov, 1970] for the case of high-dimensional phase space. Based on this theory we can conclude that the existence of homoclinic orbits $H_-(U)$ and $H_+(U)$ originating at the saddle foci ($O_+(U)$ and $O_-(U)$) with positive saddle-focus values points to the presence of complex sets of trajectories in the neighborhoods of these homoclinic orbits. These complex sets contain infinite number of unstable periodic orbits which are located outside the synchronization manifold. We should mention that Shil’nikov’s theorem does not guarantee that any of these complicated sets belong to an attractor. However, our numerical simulations and physical experiments show that, after escaping from the synchronization manifold, the system can spend some time in the neighborhood of $H_-(U)$ and $H_+(U)$, and therefore the system intermittently visits these complicated sets, see Fig. 3.

2.5. The mechanism of bursts generation

We should emphasize that all bifurcations discussed in Sec. 2 take place in the region of coupling parameters far below the value at which the robustness of synchronized oscillations can be proved by the Lyapunov function analysis. In Appendix C we prove that a Lyapunov function can be constructed for the systems (1) and (2) when the coupling is stronger than $g_{LF} \approx 21$. Therefore when $g > g_{LF}$ the synchronization of chaos is guaranteed to be robust. However our numerical simulations and physical experiment clearly demonstrate that the robustness is achieved at the values of coupling much lower than $g_{LF}$. The desynchronization outbursts seem to disappear at $g > g_c \approx 3$. Thus in our case synchronization becomes robust at the moment when the saddle fixed point embedded in the invariant chaotic set of synchronized motions becomes transversally stable. Some important critical values of the coupling parameter are shown in Fig. 5.

The reason why the transversal instability of the fixed point at the origin leads to the loss of robustness has a simple explanation. Since the saddle-focus is embedded in the invariant chaotic set of synchronized oscillations, there will be occasions when a trajectory in this set passes through a region of size $\varepsilon$ around the fixed point at the origin, no matter how small $\varepsilon$ is. This is so, provided that the system is observed for a sufficiently long time. At $g < g_c \approx 3$ the saddle focus $O_0(U)$ is transversally unstable, and there is a beam of trajectories that depart exponentially fast from the synchronization manifold. When the system evolves in the neighborhood of the fixed point $O_0(U)$, an infinitesimally small perturbation can be sufficient to relocate the state of the system from the synchronization manifold into this beam. Then the system can be carried away from the synchronization manifold up to a distance determined by the off-manifold dynamics of the system rather than by the perturbation magnitude. Thus from the moment when the saddle-focus $O_0(U)$ becomes unstable in directions transversal

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6The analytical expressions for these bifurcation values were found using Maple V symbolic computation package. We do not give the analytical expression for many of the bifurcation values of the parameter, for they are too lengthy.

![Fig. 5. The critical values of the coupling parameter $g$ and the region of robust synchronization.](image-url)
to the synchronization manifold, the robustness of synchronized chaotic oscillations is lost.

Since there are heteroclinic trajectories, $H^+(U)$ and $H^-(U)$, that connect the unstable fixed point $O_0(U)$ with the off-manifold saddle points $O_+(U)$ and $O_-(U)$, there are two beams of trajectories which, after leaving the manifold of synchronized motions, reach the neighborhood of one of the off-manifold saddle points. Each trajectory of these beams corresponds to the initiation of an outburst of desynchronization. This explains why in our numerical simulations and physical experiment the characteristic size of these outbursts is close to the distance from $O_+(U)$ and $O_-(U)$ to the synchronization manifold, see Fig. 3. We observed that the shape of the outbursts does not depend on the magnitude of perturbations introduced into the system. On the other hand, the characteristic frequency of the outbursts increases with the magnitude of perturbations. The connection between this frequency and the magnitude of perturbations can be explained by considering the properties of the escaping trajectories in the vicinity of the fixed point $O_0(U)$. Elements of this analysis can be found in [Ashwin et al., 1996].

The existence of the complicated sets discussed above can significantly increase the complexity of the system's behavior during the outbursts. A trajectory that escaped into the complicated set can demonstrate very sophisticated dynamics before it returns to the synchronization manifold. This appears to be consistent with what we observe in our numerical simulations.

2.6. "Escape points"

To illustrate the connection between the transversal instability of the fixed point $O_0(U)$ with the emergence of intermittent outbursts of desynchronized behavior, we conducted the following simulations. We numerically integrated Eqs. (1) and (2) with $g = 1.5$ (the values of other parameters were left unchanged) and uniformly distributed noise of amplitude 0.001 added to synchronization signal. The coupling was taken stronger than in the previous examples to focus on the contribution of the fixed point, and to decrease contributions of numerous transversally unstable limit cycles that are located next the homoclinic orbits. We recorded the coordinates of the points at which the trajectory punctures from the inside the 0.01-wide corridor around the synchronization manifold.

![Figure 6](image_url)

Fig. 6. Location of the "escape events" calculated with $g = 1.5$ and $\alpha = 25.361$. The crosses mark the $(x_1, x_3)$ projections of the points where the trajectory left the small corridor around the synchronization manifold in our simulations. The solid line shows two homoclinic trajectories in the driving system.

Figure 6 shows the projection of the resulting image onto $(x_1, x_3)$ plane superimposed with the homoclinic trajectories associated with the saddle-focus at the origin. We also did the calculations without noise but with a small mismatch between parameters $\alpha$ in the driving and response circuits. The resulting plot looks practically identical to the one in Fig. 6. One can clearly see that almost all "escape incidents" occur in the neighborhood of the homoclinic orbit and that they are denser in the neighborhood of $O_0(U)$.

2.7. Local transversal Lyapunov exponents

The nature of the bursting behavior can be also approached from a different prospective. It is known that even when the global transversal Lyapunov exponents are all negative, the local transversal Lyapunov exponents computed along segments of certain lengths at some locations on the attractor can be positive, see for example [Ott & Sommerer, 1994]. We have shown that in our example there are regions in the invariant chaotic set of synchronized motions where the trajectories diverge exponentially from the synchronization manifold. Therefore it makes sense to consider the regional transversal stability of the invariant chaotic set in
the synchronization manifold. This stability can be characterized by the quantity $\lambda(T) \cdot T$. Here $\lambda(T)$ is the value of the largest local transversal Lyapunov exponent and $T$ is the length of the trajectory segment along which the exponent is computed, see Appendix B. For each set of parameters, $\lambda(T) \cdot T$ is a function of $T$ and of the location in the invariant chaotic set. In regions characterized by positive values of $\lambda(T)$ the fluctuations transversal to the manifold of synchronized motions are amplified. As a result of transversal instability of the fixed point $O_0(U)$, at this fixed point the quantity $\lambda(T) \cdot T$ grows without limit as $T \to \infty$. Clearly in our case, this is also true for the homoclinic orbits in the synchronization manifold $H_0^R(U)$ and $H_0^L(U)$. Thus the quantity $\lambda(T) \cdot T$ can be arbitrarily large for the trajectories in the invariant chaotic set which are close to these homoclinic orbits. Therefore in this region even infinitesimal perturbations can be amplified to finite scale disruptions.

To illustrate this idea we conducted numerical analysis of the models (1) and (2). We studied the dependence of local transversal Lyapunov exponents on the segment length, $T$. We label each segment by the coordinates of its last point. When $T$ is small, the regions where the largest transversal Lyapunov exponent is positive cover most of the invariant chaotic set. However, small perturbations cannot be significantly amplified during the evolution along a short transversally unstable segment of trajectory. Small perturbations can result in large deviations from the synchronization manifold only when the system evolves in such regions where $\lambda(T) \cdot T$ is large positive at large $T$. When we increase $T$, the region where the largest transversal Lyapunov exponent is positive shrinks and at large $T$ surrounds tightly a part of the homoclinic trajectories. Figure 7 shows the distribution of the maximum local transversal Lyapunov exponent computed with $T = 10.0$ in the projection onto the plane $(x_1, x_3)$ of the driving variables. In this projection the invariant chaotic set of synchronized motions is identical to the attractor of the driving system. It is easy to see that the region near the one-dimensional unstable manifold $W^u_1[O_0(X)]$ of the saddle-focus $O_0(X)$ is indeed characterized by positive values of the local transversal Lyapunov exponents.

To conclude this section, we should emphasize that the existence of simple homoclinic orbits is

![Image](image.png)

**Fig. 7.** Distribution of local transversal Lyapunov exponents computed for the systems (1) and (2) along trajectories with $T = 10.0$ and $g = 1.5$. The blue dots mark the locations where the local transversal Lyapunov exponents are negative while the red color shows the region in the invariant chaotic set where the exponents are positive. A detailed analysis of this distribution indicates that the larger values of local transversal Lyapunov exponents are encountered closer to the homoclinic trajectories.
not crucial for the fact of destabilization of synchronized oscillations by small perturbations. The existence of the homoclinic orbits allowed us to construct the skeleton of trajectories that we used for the analysis of the complicated motions appearing outside the synchronization manifold when the synchronized chaos ceases to be robust. The presence of such homoclinic orbits may (and, in our example, does) affect the characteristic frequency of the desynchronization outbursts. The presence of transversally unstable saddle in the immediate vicinity of the attractor in the manifold of synchronized motions by itself guarantees the loss of robustness and destabilization of the regime of synchronized chaotic oscillations in the presence of perturbations. However, once the robustness of synchronized oscillations is lost, the existence of simple homoclinic trajectories leads to a much more powerful amplification of perturbations because the trajectory that passes once near the origin is likely to stay near the homoclinic orbit and, thus, likely to re-enter the region of high transversal instability in a very near future. This can lead to a sequential amplification of perturbations time after time and result in a higher frequency of large scale outbursts away from the synchronization manifold.

3. Loss of Robustness Facilitated by Transversally Unstable Periodic Orbits

In the previous section we demonstrated that a transversally unstable fixed point embedded in the invariant chaotic set of synchronized motions is, in some cases, responsible for the loss of stability of synchronized oscillations in the presence of perturbations. However the mechanism described there is not general, since in many examples the invariant chaotic set of synchronized motions does not contain any fixed points. At the same time, it is quite a general situation that saddle periodic orbits are embedded into the invariant chaotic set. In such cases the bursting behavior can be caused by transversal instability of some of these periodic orbits.

Consider a saddle periodic orbit embedded in the attractor of the driving system. In the combined phase space $U$ there is a corresponding saddle periodic orbit that lies in the manifold of synchronized motions $x = y$. Depending on the coupling strength, this orbit can be either stable or unstable in directions transversal to the manifold. The latter can (and does) occur even in the cases when the transversal Lyapunov exponents computed along a typical trajectory in the invariant chaotic set of synchronized chaotic motions are all negative [Hunt & Ott, 1996]. Since the saddle periodic orbit is embedded in the attractor of the driving system, the full system during its evolution in the manifold of synchronized motions can come very close to the corresponding periodic orbit in the full phase space. These events may occur very seldomly, which results in their small contribution into the global transversal Lyapunov exponents. Consider the situation when the periodic orbit is transversally unstable. Then, as in the case with transversally unstable fixed points, there is a beam of trajectories that exponentially depart from the synchronization manifold. Therefore small perturbations can produce outbursts of desynchronization in a manner similar to the one described in the previous section. In this section we shall briefly consider the bifurcations associated with the loss of the transversal stability by some of periodic orbits in the synchronization manifold.

3.1. Numerical analysis

To illustrate how the existence of transversally unstable periodic orbits in the invariant chaotic set of synchronized motions can lead to the loss of robustness, we start with numerical integration of Eqs. (1) and (2) with $\alpha = 23$. We recorded the maximum of the difference $x_1(t) - y_1(t)$ as a function $\Delta(g)$ of the coupling parameter $g$ for two cases: Without any noise or parameter detuning and with noise of small amplitude ($0.01\%$ of the amplitude of $x_1$) occasionally added to the variables during the integration process. This function is shown in Fig. 8.
We observed that when the coupling was less than $g_{cr} \approx 0.1$ the function $\Delta(g)$ is practically the same for both cases. However when $g$ was larger than $g_{cr}$, the system without noise would come to the manifold of synchronized motions and $\Delta(g)$ was zero, while the system perturbed by noise departed from the synchronization manifold to quite significant distances until $g > g_s \approx 0.3$, see Fig. 8.

In this case, the appearance of disruptions of synchronization in certain intervals of the coupling parameter can be explained by the amplification of small perturbations near the transversally unstable saddle limit cycles. Let us consider how the bifurcations of simple saddle periodic orbits in the synchronization manifold can be connected with the loss of robustness of synchronized oscillations. The attractor in the driving system with $\alpha = 23$ is shown in Fig. 9(a). Figure 9(b) shows period-1 (p1) and period-2 (p2) saddle limit cycles.

The numerical stability analysis for the period-1 orbit in the six-dimensional phase space of driving and response circuits revealed the following. The limit cycle in the manifold of synchronized motions is transversally stable for $g > g_2(p1) \approx 0.3$. At $g_2(p1)$ the period-1 cycle undergoes the period doubling bifurcation, becomes transversally unstable, and the doubled period limit cycle (2p1) is formed outside the manifold of synchronized motions, see Fig. 10. This newly born cycle suffers the next period doubling bifurcation at $g_4(p1) \approx 0.11$. From the moment when the period-1 saddle limit cycle becomes transversally unstable, there is a possibility that, when the system comes to the vicinity of this cycle, small perturbations can relocate the state of the system into the beam of trajectories that depart from the synchronization manifold. This will result in the appearance of outbursts of desynchronized behavior.

Some other limit cycles lose transversal stability in a very similar manner and have similar bifurcation sequences associated with them. For instance, period-2 limit cycle suffers the doubling bifurcation at $g_{2}(p2) \approx 0.31$ at which point the period-4 cycle (2p2) is formed outside the synchronization manifold, see Fig. 10(c). When the coupling is weaker than $g_{2}(p2)$, the period-2 limit cycle in the manifold of synchronized motions becomes the core of another region of transversal instability.

The sequence of the period doubling bifurcations associated with the limit cycles leads to the appearance of a set of unstable limit cycles outside the manifold of synchronized motions. Since the cycles which appear as a result of these bifurcations are of a saddle character, they can lead to the formation of a homoclinic structure outside the manifold. The existence of such structure significantly increases the duration of the transients which the system goes through once it escapes from the manifold of synchronized motions.

3.2. Experimental analysis

To experimentally study the bifurcations of periodic orbits which lead to the loss of robustness of synchronized motions, we investigated the behavior of two nearly identical coupled chaotic circuits (see Fig. 1) with parameters set in such a way that the attractor of the driving circuit is asymmetric, similar to the one in Fig. 9(a). We analyzed the experimental chaotic data produced by the driving
circuit to reconstruct the waveforms of period-1 and period-2 saddle limit cycles which are embedded in the chaotic attractor. In all three cases that we considered, the response circuit was driven either chaotically by the signal from the driving circuit or by periodic signals corresponding to the period-1 and period-2 limit cycles in the driving circuit. The periodic driving signals were generated by a computer using the analog output of the data acquisition board (AT-MIO-16X, by National Instruments). The results of the experiment are summarized in the Table 1.

Driving the response circuit with the periodic signals, we observed up to four sequential period doubling bifurcations for the period-1 and period-2 orbits and the formation of invariant chaotic sets that terminates the period doubling sequence in each case. The parameters at which the first three period doubling bifurcations in each sequence take place are summarized in the Table 1. These bifurcations occur in accordance with the Feigenbaum scenario and result in the appearance of invariant chaotic sets outside the synchronization manifold. The last row in the Table 1 contains the bifurcation values of the resistor $R_c$ where the period doubling sequences result in the formation of the invariant chaotic sets. So far we restricted the evolution of the driving system to a saddle periodic orbit. If we now consider the unrestricted dynamics of coupled systems, these bifurcation sequences will be the sequences of period doubling bifurcations of saddle periodic orbits which result in the appearance of saddle chaotic sets of the Feigenbaum type. Therefore, these bifurcations present a mechanism of

<table>
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<th>Experimental Observed Bifurcations of the Limit Cycles Under Periodic Driving</th>
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<td>Driving by Period-1 Waveform</td>
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<td>--------------------------------</td>
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<tr>
<td>First period doubling</td>
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<td>Second period doubling</td>
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Fig. 10. The $(x_1, y_1)$ projections of the limit cycles formed in the sequence of period doubling bifurcations associated with transversal destabilization of period-1 and period-2 saddle limit cycles: (a) the 2p1 cycle at $g = 0.14$ after the first period doubling bifurcation of the period-1 cycle, (b) the 4p1 cycle at $g = 0.08$ formed after 2p1 cycle undergoes the period doubling bifurcation, (c) the periodic orbit at $g = 0.2$ that appears after the period-2 limit cycle in the manifold becomes unstable. (Note: The original period-1 and period-2 limit cycles project onto the plane $(x_1, y_1)$ as the diagonal $x_1 = y_1$.)
formation of complicated sets of unstable orbits outside the synchronization manifold.

Since in the experimental setup there is always noise and parameter mismatch, the loss of robustness of the synchronized regime of chaotic oscillations results in outbursts away from the synchronization manifold. In our experiment this bursting behavior becomes noticeable at the value of the coupling resistance $R_c \approx 1.7$ kΩ. As one can see from the Table 1, the critical value of the coupling below which we begin to see the bursting behavior in the coupled systems is approximately equal to the bifurcation values of the coupling parameter where the period-1 and period-2 cycles in the manifold of synchronized motions become unstable. Thus, our experimental observations support the connection between transversally unstable saddle periodic orbits and the loss of robustness of synchronized oscillations.

The role of periodic orbits in the process of desynchronization was analyzed in [Heagy et al., 1995]. There it was pointed out that when the system occasionally jumps off the synchronization manifold, this usually happens in the neighborhoods of periodic orbits in the attractor of synchronized motions. Our conclusions seem to be quite consistent with observations of Heagy et al.

4. Summary and Discussion

In this paper we considered bifurcation mechanisms which accompany the breakdown of synchronized chaos in the presence of small perturbations, such as noise or parameter mismatch. We showed that chaos synchronization is extremely sensitive to such perturbations if there are transversally unstable invariant limiting sets (such as saddle fixed points or limit cycles) embedded in the invariant chaotic set of synchronized motions. The existence of such invariant limiting sets results in intermittent outbursts of unsynchronized behavior, if perturbations are present. Distributions of "escape points" and of local transversal Lyapunov exponents reflect the connection between such transversally unstable limiting sets and the desynchronization outbursts.

We analyzed the bifurcations that result in formation of such transversally unstable invariant limiting sets. It was shown that these sets can be formed as a result of bifurcations of fixed points and/or saddle limit cycles embedded in the invariant chaotic set in the synchronization manifold. These bifurcations produce new fixed points and/or limit cycles outside the synchronization manifold (off-manifold fixed points and limit cycles).

We inspected the characteristic motions emerging outside the synchronization manifold that determine the structure of trajectories segments during the desynchronization outbursts. Analyzing the bifurcations of off-manifold limit cycles, we found that period doubling sequences associated with these limit cycles is a possible mechanism for the formation of complicated invariant sets that appear outside the synchronization manifold. The complication of the characteristic motions outside the manifold can increase the duration of intermittent outbursts and eventually results in complete loss of synchronization.

One more interesting conclusion can be made from the analysis presented in Sec. 3. It appears that in the regime of synchronized chaotic oscillations, each limit cycle embedded in the attractor of the driving system maps into a single periodic orbit in the state space of the response system. When a limit cycle embedded in the invariant chaotic set of synchronized motions undergoes a period doubling bifurcation, there are two periodic orbits in the state space of the response which correspond to a single limit cycle in the driving system. At the same time, it was shown that after such bifurcation, the regime of synchronized oscillations ceases to be robust. The property that each limit cycle embedded in the attractor of the driving is mapped into a single periodic orbit in the state space of the response system can be considered as one of the conditions necessary for the robustness of identical chaotic oscillations.

Finally, we would like to note that the bifurcation mechanisms considered here can be important not only for the problems of chaos synchronization but also for some other problems where one deals with the stability of invariant chaotic sets located in invariant sub-manifolds. For instance, the mechanisms considered in this paper may be relevant to the analysis of systems with on-off intermittency [Platt et al., 1993].

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References


Appendix A
Electronic Circuits Implementation

To study the stability issues in real physical systems we built two coupled electronic circuits, each consisting of a nonlinear converter \( N \) and linear feedback (Fig. 1). Depending on the parameters of the linear feedback and the shape of the nonlinearity, this circuit can generate various regimes of chaotic oscillations. To make the paper self-contained, we shall describe the implementation of the circuit.

In the experiments we used a nonlinear converter whose diagram is shown in Fig. 11. The converter transforms the input voltage \( x \) into the output which has nonlinear dependence \( F(x) \equiv \alpha f(x) \), where the parameter \( \alpha \) characterizes the gain of the converter at \( x = 0 \). The nonlinear function \( f(x) \) is an odd function: \( f(x) = -f(-x) \). It was empirically found that the shape of the nonlinearity produced by the converter, \( N \), can be very well approximated by the following function

\[
f(x) = \text{sign}(x) \left( \frac{d(f_1(x) - a)^2 + c - a}{d} \right) \tag{A.1}
\]

where

\[
f_1(x) = \begin{cases} |x| & \text{if } |x| \leq a \\ -q(|x| - p) & \text{if } a < |x| \leq b \\ -a & \text{if } |x| > b 
\end{cases}
\]

\[
d = \frac{a^2 - c}{a^2}, \quad q = \frac{2a}{b-a}, \quad p = \frac{b+a}{2}.
\]

When the values of the parameters \( a, b \) and \( c \) are chosen to be equal to 0.5, 1.8 and 0.03 respectively, the function (A.1) fits the actual nonlinearity of the converter with \( \approx 2\% \) accuracy.

The linear feedback contains the low pass filter \((RC')\) and the resonant circuit \(rLC\). In the experiments, the parameters of the feedbacks in drive and response circuits were set to be as close as possible and equal, \( C \approx 342 \text{nF}, C' \approx 225 \text{nF}, L \approx 145 \text{mH}, r \approx 348 \text{Ω} \) and \( R \approx 4.97 \text{kΩ} \).

We denote the voltages across the capacitors \( C_1 \) and \( C'_1 \) by \( x_1 \) and \( x_3 \) correspondingly. The similar voltages in the response system we call \( y_1 \) and \( y_3 \). Since the nonlinear converters consume no input currents, it follows from the current balance at \( C_1 \) and \( C_2 \) that the currents \( J_1(t) \) in the driving circuit and \( J_2(t) \) in the response circuit can be written as

\[
J_1 = C_1 \frac{dx_1}{dt}
\]
\[
J_2 = C_2 \frac{dy_1}{dt} + \frac{1}{R_c} (y_1 - x_1)
\]

At the same time

\[
x_3 - x_1 - L \frac{dJ_1}{dt} = r_1 J_1
\]
\[
y_3 - y_1 - L \frac{dJ_2}{dt} = r_2 J_2
\]

Finally, the current balance at capacitors \( C'_1 \) and \( C'_2 \) yields

\[
J_1 = -C'_1 \frac{dx_3}{dt} + \frac{1}{R_1} (\alpha_1 f(x_1) - x_3)
\]
\[
J_2 = -C'_2 \frac{dy_3}{dt} + \frac{1}{R_2} (\alpha_2 f(y_1) - y_3)
\]
Let us assume that all electronic components in the drive and response circuits are identical: \( C_1 = C_2 = C, \ C'_1 = C'_2 = C', \ L_1 = L_2 = L, \ \tau_1 = \tau_2 = \tau, \ R_1 = R_2 = R \). Introducing new time \( t_{\text{new}} = t/(LC)^{1/2} \) and normalizing the currents \( J_{1,2}(t) \) through the inductors \( L_{1,2} \) by the factor \((L/C)^{1/2}\) and calling the normalized values \( x_2 \) and \( y_2 \) correspondingly, we obtain the equations (1) and (2) and the connection between the parameters in these equations with the real parameters of the circuits in the following form

\[
\delta = \tau(C/L)^{1/2} \approx 0.534,
\gamma = (LC)^{1/2}/(RC') \approx 0.2, \quad \sigma = C/C' \approx 1.52
\]

These values of the parameters were used in the analytical considerations and in the numerical simulations which we discuss in the Secs. 2 and 3.

Appendix B

Transversal Lyapunov Exponents

In this Appendix we shall describe the procedure for computation of transversal Lyapunov exponents for the invariant chaotic set in the synchronization manifold.

Let us consider the perturbations transversal to the synchronization manifold \( x = y \) in the systems (1) and (2). These perturbations can be written as \( \xi(t) = x(t) - y(t) \). The linearized equations that describe the evolution of these perturbations close to the synchronization manifold are

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 - g\xi_1 \\
\dot{\xi}_2 &= -\xi_1 - \delta\xi_2 + \xi_3 \\
\dot{\xi}_3 &= \gamma \left( \frac{df(z)}{dz}|_{z=x_1(t)} \right) \xi_1 - \xi_3 - \sigma\xi_2
\end{align*}
\]

where \( x_1(t) \) corresponds to the motion along a typical chaotic trajectory in the attractor of the driving system (1).

The Lyapunov exponents for this system are computed as follows. Consider a vector \( \xi(t_i) \) at \( t = t_i \). In the course of evolution of system (B.1) over time \( t_f = t_i + \Delta t \) this vector maps into the vector \( \xi(t_f) \). Since system (B.1) is linear, these two vectors are connected by the linear transformation \( \hat{K}(x(t_i), \Delta t): \xi(t_f) = \hat{K}(x(t_i), \Delta t)\xi(t_i) \). We select a set of initial vectors \( \xi(t_i) \), find (by numerical integration) the corresponding set of final vectors \( \xi(t_f) \) and solve the standard problem of linear algebra to find the matrix \( \hat{K}(x(t_i), \Delta t) \).

To compute the Lyapunov exponents along a trajectory segment of the length \( T \), this segment is divided into \( N \) sub-segments of the length \( \Delta t \).\(^7\) The matrix \( \hat{K}(x(t_i), \Delta t) \) is computed for each segment. The matrix \( \hat{K}(x(t_i), T) \) defined by \( \xi(t+T) = \hat{K}(x(t), T)\xi(t) \) can be computed in the following way:

\[
\hat{K}(x(t), T) = \prod_{n=0}^{N-1} \hat{K}(x(t+n\Delta t), \Delta T)
\]

The eigenvalues \( \mu_i(x(t), T) = \frac{1}{\Delta t} \ln \mu_i(x(t), T) \). These are the local Lyapunov exponents since they are computed along finite segments of trajectories. The limits of these numbers at \( T \to \infty \) is what we called in this paper the global Lyapunov exponents: \( \lambda_i = \lim_{T \to \infty} \lambda_i(x(t), T) \). The existence of this limit for motions on a chaotic invariant set is guaranteed by the Oseledec theorem [Oseledec, 1968]. In reality, we, of course, cannot numerically compute the exponents for infinite \( T \). Therefore the global Lyapunov exponents are estimated as the values of \( \lambda_i(x(t), T) \) at which they appear to saturate at large \( T \). Some aspects of convergence are discussed in [Abarbanel et al., 1993; Brown et al., 1994].

Note that the same calculation with \( g = 0 \) in (B.1) produces the values of Lyapunov exponents for the chaotic attractor of the driving system.

Appendix C

Lyapunov Function Analysis

In some intervals of the coupling parameter values, the transversal stability of the manifold of synchronized motions can be proved by means of the Lyapunov function method. In this appendix we present the Lyapunov function analysis of the stability of the manifold in the systems (1) and (2).

\(^7\)This division is done only to avoid certain computational difficulties (such as under- and over-flow and error accumulation).
The equations for the perturbations transversal to the synchronization manifold (3) can be written in the following form:

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 - g\xi_1 \\
\dot{\xi}_2 &= -\xi_1 - \delta \xi_2 + \xi_3 \\
\dot{\xi}_3 &= \gamma [\alpha \Phi(x_1(t), y_1(t)) \xi_1 - \xi_3] - \sigma \xi_2
\end{align*}
\]  
(C.1)

where the variables $\xi_i$ are the components of the vector $\xi = x - y$ and $\Phi(x_1(t), y_1(t)) = \frac{f(x_1(t)) - f(y_1(t))}{x_1(t) - y_1(t)}$.

Consider the non-negative quadratic function

\[
V(\xi_1, \xi_2, \xi_3) = \frac{1}{2} \left( \xi_1^2 + \xi_2^2 + \frac{1}{\sigma} \xi_3^2 \right)
\]  
(C.2)

which is equal to zero only in the synchronization manifold, $\xi = 0$. For the system (C.1) the derivative of the function $V$ with respect to time can be written as

\[
\frac{dV}{dt} = -\left( g - \frac{\gamma \alpha^2 \Phi(x_1(t), y_1(t))^2}{4\sigma} \right) \xi_1^2 - \xi_2^2
- \frac{\gamma}{\sigma} \left( \xi_3 - \frac{\alpha \Phi(x_1(t), y_1(t))}{2} \right) \xi_1^2.
\]  
(C.3)

Since the values of the parameters $\delta$, $\gamma$ and $\sigma$ are positive, the derivative (C.3) will be negative for all values of the perturbations (except for $\xi = 0$) if the following condition is satisfied

\[
g > g_{LF} = \max \left\{ \frac{\gamma \alpha^2 \Phi(x_1(t), y_1(t))^2}{4\sigma} \right\}.
\]  
(C.4)

It follows from the analysis of the function (A.1) that $\Phi(x_1(t), y_1(t))^2 \leq 1$ for all values of $x_1(t)$, $y_1(t)$. Therefore, the value of $g_{LF}$ is bounded from above and the condition (C.4) can be satisfied for sufficiently strong coupling. When the condition is satisfied, the function (C.2) is the Lyapunov function of the system (C.1) and all trajectories started outside the manifold are attracted by the manifold of synchronized motions.

For the parameters values of the circuits considered in Sec. 2 the value of $g_{LF}$ is 21.16. This value of the coupling parameter is about seven times larger than required for the robust synchronization of chaos, see Fig. 5.