Threshold synchronization of chaotic relaxation oscillations

N.F. Rul'kov and A.R. Volkovskii
Lobachevskii State University, 23 Gagarin Avenue, 603600 Nizhny Novgorod, Russian Federation

Received 20 May 1993; accepted for publication 14 June 1993
Communicated by V.M. Agranovich

A new type of synchronized chaos conditioned by the threshold synchronization of relaxation oscillators with chaotic behaviour is studied experimentally. It is shown that for a certain parameter ratio the pulses generated by the chaotic oscillator may be synchronized by periodic and chaotic pulse sequences generated by the drive oscillator. It is found that the threshold synchronization regime enables one to detect modulation of chaotic signals.

1. The synchronization phenomenon in systems with chaotic behaviour is known to lead to different nonlinear effects. It was shown in refs. [1—5] that small time-dependent perturbations of a parameter of a system with chaotic behaviour can eliminate chaos as a result of stabilization of periodic motion near the unstable limit cycle which is contained in the chaotic attractor. It is also known that in the case of coupled systems with individual chaotic behaviour the transition to synchronization may give rise to a regime of synchronized chaotic oscillations [6—14]. It is obvious that like in the case of synchronization of periodic oscillations these phenomena may play a significant role in the behaviour of many physical systems as well as in different applications.

Although synchronization in different chaotic systems was described in quite a few papers, the features of this phenomenon in chaotic relaxation systems still need further investigation. The alternation of fast motion (pulsation) and slow evolution is typical for relaxation systems. The transition from slow to fast motion occurs at the moments when the system reaches a certain threshold state. Immediately before the threshold the relaxation system is extremely sensitive to small external perturbations which may cause a forced transition to fast motion. This property is widely used in electronics for synchronization of periodic pulse oscillators driven by weak periodic synchronization signals. This mechanism of synchronization is known as threshold synchronization [15]. A similar type of synchronization phenomenon was considered for the problem of synchronization of pulse-coupled biological periodic oscillators [16,17].

This paper presents some results of theoretical and experimental investigation of threshold synchronization in relaxation systems obtained by modeling with special electronic circuits.

The paper includes a brief description of the electronic circuit used for modeling the relaxation chaotic system (section 2), investigation of the effect of chaos suppression in this system by applying weak periodic pulses (section 3), and synchronization of chaos in a pair of unidirectionally coupled relaxation chaotic systems (section 4). Examples of applications of synchronized chaotic relaxation oscillators are discussed in section 5.

2. The algorithm for the performance of a chaotic relaxation circuit can be realized within a block scheme shown in fig. 1a. Slow evolution of the system corresponds to an increasing voltage $u_2$ on the capacitor $C$ when charged. When this voltage reaches the threshold level, $u_2(t) = 0$, the comparator switches to another state and the pulse oscillator generates a pulse $d_n$ at time $t_n$, a sample circuit controlled by this pulse provides recharging of the capacitor $C$ to the voltage $u_2 = -\sigma u_1(t_n)$ and sets the nonlinear curve generator to the initial state $u_1 = 0$. After that, the capacitor $C$ is charged from a current source $I_0$ and,
when its voltage, $u_2$, reaches the threshold level again, the pulse oscillator generates the next pulse, $d_{n+1}$. So the circuit produces a pulse sequence $d(t)$ with time intervals between the pulses determined as

$$T_{n+1} = \alpha u_1(t_n) + T_p, \quad u_1(t_n) = F(T_n),$$

where $\alpha = \sigma C / I_0$, $F(T)$ is the output voltage of a nonlinear curve generator. We use a triangular type function $F(t)$ (figs. 1b and 1c) with the parameters $u_{\text{min}} = 2$ V and $u_{\text{max}} = 7$ V. The slopes of the function $F(T)$ on increasing and decreasing intervals were assumed to be the same and equal to $\beta = |\beta'|$, where $\beta' = dF/dT$. $\alpha$ and $\beta$ were used as control parameters.

A model of a chaotic relaxation oscillator (1) is a 1D map. The trajectories of this map can be easily studied by means of the plot presented in fig. 1c. If the parameters $\alpha$ and $\beta$ are chosen so that $\alpha\beta > 1$ all trajectories of the map come to a strange attractor that is bounded by $T_{\text{min}} = \alpha u_{\text{min}}$ and $T_{\text{max}} = \alpha u_{\text{max}}$ and characterized by a positive Lyapunov exponent $\lambda = \ln(\alpha\beta) > 0$. Choosing $F(T)$ as a piecewise linear function one can design an oscillator which provides a smooth variation of the characteristics of the chaotic oscillations in a broad parameter region.

3. A strange attractor mapped by (1) contains one or more unstable fixed points $O_i$ which are located at the intersections of curves 1 and 2 (see fig. 1c). The points correspond to the existence of the regime of periodic oscillations of the oscillator in the form...
of a pulse sequence with period $T^{(i)}$. It is obvious that such regimes are unstable due to the instability of the fixed points.

Consider the weak periodic forcing of a chaotic oscillator. In order to study this influence in an experiment, an external periodic pulse signal $d_{dr}(t)$ with amplitude $E$ and period $T_{dr}$ is applied to the input “Synch”. The pulses $d_{dr}$ are added to the voltage $u_2(t)$. If at the moment of the pulse action the voltage $u_2$ is close enough to the threshold level, i.e. the conditions $0 > u_2 > -E$ are satisfied, then the electronic circuit will generate a pulse $d$ in step with $d_{dr}$. Otherwise the pulse $d_{dr}$ has no influence on the operation of the circuit.

In the case of small amplitude $E$ ($E < T_{max}/\alpha$), the signal $d_{dr}$ provides synchronization if the parameters of the signal satisfy the following conditions,

$$\alpha E/\sigma > (T_{dr} - T^{(i)}) (\alpha \beta^{(i)} - 1) > 0,$$

where

$$\beta^{(i)} = \frac{dF}{dT} \bigg|_{\tau = T^{(i)}}.$$

Figure 2 shows the threshold synchronization zones in the plane of the parameters $(T_{dr}, E)$ when the parameters of the oscillator are like in fig. 1c. One can see from fig. 2 that even for very weak periodic forcing chaos can be suppressed and a regime of periodic synchronized oscillations will be realized. Mention should be made that threshold synchronization is possible if the period of weak forcing is close enough to the period $T^{(i)}$.

Note that, besides the unstable fixed points $O_n$ map (1) contains an infinite number of unstable limit cycles. These periodic motions can also be synchronized by periodic forcing at higher order synchronisms. This indicates the existence of a complex fractal structure of the synchronization zones for periodic oscillations of different types. This complex structure is not shown in fig. 2.

4. It should be emphasized that the solutions of map (1) include open unstable trajectories which belong to the strange attractor. These motions correspond to the oscillatory regime of the system in which the output signal $d(t)$ looks like a chaotic sequence of pulses. It seems interesting to investigate the threshold synchronization along one of these trajectories if the synchronizing signal is generated by the same chaotic oscillator.

In an experimental investigation of threshold synchronization by external chaotic forcing, the output pulses $d_{dr}$ with amplitude $E$ generated by a driving chaotic oscillator were led to the input “Synch” (see fig. 1a) of the synchronizing chaotic oscillator. The parameters $u_{min}$, $u_{max}$ and $T_p$ of the driving and synchronizing oscillators were set to be close and fixed. The parameters $\alpha_{dr}$, $\beta_{dr}$, $\alpha_s$, $\beta_s$ and the normalized amplitude $\epsilon = E/\alpha u_{max}$ of the pulses $d_{dr}$ were taken as control parameters. Here and in the following the subscripts “dr” and “s” denote the parameters of driving and synchronizing oscillators, respectively.

The synchronization regions in the parameter plane $(\alpha_s$, $\epsilon)$ with fixed values of $\alpha_{dr}$ and $\beta_{dr} = \beta_s$ are shown in fig. 3a. The diagrams in fig. 3a show that locking is possible if $\alpha_s > \alpha_{dr}$ which indicates that, for threshold synchronization, each synchronizing pulse $d_{dr}$ should appear before the voltage $u_{2s}$ reaches the threshold level. The loss of synchronization with increasing $\alpha_s$ is caused by the fact that the voltage $u_{2s}$ is so small that, at the moment the pulse $d_{dr}$ appears, the threshold level cannot be reached. It is quite easy to show that if the parameters $u_{max}$, $u_{min}$, $T_p$ and $\beta$ are the same in both oscillators, the synchronization zone is defined by the following conditions,

$$\alpha_s > \alpha_{dr}, \quad \epsilon > 1 - \alpha_{dr}/\alpha_s.$$  

The spread in the parameters leads to narrowing of the synchronization zones which vanish when $\epsilon = \epsilon_{min} > 0$. 

Fig. 2. Synchronization zones of the chaotic oscillator forced by a periodic pulse signal.

334
Note that, unlike the periodic oscillations for which only one parameter (period) is essential, the synchronization of chaotic oscillations is determined by a few parameters. Consequently the threshold synchronization of chaotic signals is possible only when all parameters of the oscillators are sufficiently close. For example $\beta$ can be considered as an additional control parameter. Analysis of map (1) shows that when the correspondent parameters $u_{\text{min}}, u_{\text{max}}$ and $T_p$ are equal but $\beta_{\text{tr}} \neq \beta_s$, synchronization can be realized if

$$
\epsilon > 1 - \alpha_{\text{dr}}/\alpha_s = \epsilon_{cr}, \quad |\beta_s - \beta_{\text{tr}}| < (\epsilon - \epsilon_{cr})/\alpha_s,
$$

$$
|\beta_s - \beta_{\text{tr}}| < \frac{u_{\text{min}} \epsilon_{cr}}{u_{\text{max}} \alpha_{\text{dr}}}. \quad (4)
$$

The region of synchronization in the parameter plane $(\beta_s, \epsilon)$ is presented in fig. 3b for some values of $\beta_{\text{tr}}$ and fixed $\alpha_s$ and $\alpha_{\text{dr}}$. One can see that locking occurs only in a narrow zone of $\beta_s$ close to $\beta_{\text{tr}}$.

5. The threshold synchronization mechanism is quite general for self-oscillatory relaxation systems. For example, there are many studies devoted to synchronization of periodic relaxation oscillations by external periodic stimulation in biological systems (excitable membranes, cardiac models, breathing systems, etc.) [18--21]. It is known (see, for instance, refs. [19,22]) that under certain conditions the behaviour of biological systems becomes chaotic. Therefore it seems natural to suppose that the effects related to threshold synchronization are also intrinsic to the dynamics of biological chaotic oscillators.

There are a number of works [23–25] in which chains of coupled relaxation oscillators are used as earthquake fault models. The dynamics of these models leads to the appearance of spatio-temporal chaos caused by irregular formation and disintegration of domains which consist of several neighbouring oscillators near the threshold state. If the model takes into account the interaction between the domains conditioned by sound wave generation, then the threshold synchronization regime can provide simultaneous generation of pulsations at different points in space.

Different ways of employing systems with chaotic dynamics in communication systems for secret information were suggested in a number of recent papers [26–28]. The threshold synchronization seems also to be useful in realizing this idea. As it was mentioned above, the pulse oscillator described in section 2 enables one to change the characteristics of a chaotic signal smoothly in a broad parameter region. This property can be used for modulation, the regime of threshold synchronization for demodulation of chaotic signals.

It can be seen that the voltage $u_{2s}$ never reaches the threshold level if the oscillator operates in the regime of threshold synchronization. The mean value of this voltage at the moment a synchronizing pulse appears, $\bar{u}$, depends on the parameter difference (see fig. 4a). This dependence has a linear part where the voltage $\bar{u}$ is proportional to $\alpha_{\text{dr}}$. This property can be used for demodulation. The process of demodulation is shown in fig. 4b for the sinusoidally modu-
References