Images of synchronized chaos: Experiments with circuits

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Synchronization of oscillations underlies organized dynamical behavior of many physical, biological and other systems. Recent studies of the dynamics of coupled systems with complex behavior indicate that synchronization can occur not only in case of periodic oscillations, but also in regimes of *chaotic* oscillations. Using experimental observations of chaotic oscillations in coupled nonlinear circuits we discuss a few forms of cooperative behavior that are related to the regimes of synchronized chaos. This paper is prepared under the request of the editors of the special focus issue of *Chaos* and contains the materials for the lecture at the International School in Nonlinear Science, ''Nonlinear Waves: Synchronization and Patterns,'' Nizhniy Novgorod, Russia, 1995. The main goal of the paper is to outline the collection of examples that illustrate the state of the art of chaos synchronization.
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One of the most significant properties of oscillations generated by nonlinear dynamical systems is their ability to be synchronized. The synchronization of the oscillations can be achieved due to either an external forcing or couplings between the systems. Although synchronization is originally referred to systems with periodic oscillations, recent studies show that phenomenon of synchronization extends to cases of irregular, chaotic oscillations generated by dynamical systems. The growing interest in systems with chaotic dynamics and their applications in different fields of science bring new issues to the synchronization theory. Despite the fact that many recent publications discuss examples of synchronized chaos, the general framework of this phenomena has not been quite developed. The main point of this paper is to present examples of chaos synchronization that illustrate the modern state of this framework. In order not to be abstract we consider the cases of chaos synchronization that we observed in experiments with electronic circuits. We also discuss mechanisms responsible for the onset of synchronization.

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I. INTRODUCTION

Synchronization of oscillations is a well-known nonlinear phenomenon that is frequently encountered in nature. The ability of nonlinear oscillators to synchronize each other is a basis for the explanation of many processes of nature and, therefore, synchronization plays a significant role in science. Numerous applications of the synchronization in me-

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chanics, electronics, communication, measurements, and in many other fields have shown that the synchronization is extremely important in engineering.

Usually, synchronization is understood as the ability of coupled self-excited oscillators with different frequencies to switch their behavior from the regime of independent oscillations characterized by beats to the regime of cooperative stable periodic oscillations, as the strength of the coupling is increased. As a result of synchronization, the oscillators change their frequencies in a such way that these frequencies become identical or related via a rational factor. Depending upon the properties of oscillators considered, there are different explanations as to why these oscillators synchronize. For instance, one can distinguish between the mechanism for synchronization of relaxation oscillators that produce sharp pulses and the mechanism for synchronization of the oscillators that generate smooth waveforms (the main features of these mechanisms are outlined in Appendix A). However, one should keep in mind that the separation of these mechanisms is not precise, and they can be considered as limiting cases of a general mechanism, where resonances and actions of dissipative forces are very important.

A. Periodic motions and synchronization

The first nonlinear theory for synchronization of quasi-harmonic oscillations is due to van der Pol.[1] Now, there are many books and reviews that discuss different aspects of the synchronization theory for periodic oscillators. Studying Refs. 2–8 can be very useful for understanding the main features of the onset of synchronized behavior.

Despite the complexity of mechanisms involved in the process of the onset of synchronization, the phenomenon of synchronization of periodic oscillations has a transparent geometrical interpretation. Indeed, in the phase space of an autonomous system the image of the periodic oscillations is a stable limit cycle. Behavior of a network of n coupled periodic oscillators depends on the parameters of couplings among the oscillators. In the phase space of the network with zero couplings, the post-transient oscillations will correspond to trajectories that fill the surface of a stable n-dimensional torus. The phenomenon of synchronization among periodic oscillators is understood as the ability of the network to switch its behavior from quasiperiodic oscillations, associated with aperiodic motions on a stable torus, to periodic oscillations, as the strength of the couplings between the oscillators is increased. Therefore, the image of the synchronized periodic oscillations is again a stable limit cycle, but now this limit cycles is in the joint phase space of the network of coupled oscillators.

B. Synchronization of chaotic oscillations

The progress in studies of nonlinear dynamical system that was achieved in the past two decades significantly extended the notion of oscillations in nonlinear systems. It was shown that the post-transient oscillations in dynamical systems can be associated not only with regular behavior such as periodic or quasiperiodic oscillations, but also with chaotic behavior. At the same time it was shown that some of the ideas of synchronization can be extended for description of particular types of cooperative behavior in coupled systems with chaotic dynamics. For example, Fujisaka and Yamada have demonstrated that two identical systems with chaotic individual dynamics can change their behavior from uncorrelated chaotic oscillations to identical chaotic oscillations, as the strength of the coupling between the systems is increased.[9]

Recent ideas of employing the synchronization of chaos in different applications resulted in intensive studies in this field of nonlinear dynamics. Numerous papers, published in the past six years, employed the ideas of synchronized chaos for different techniques of communications with chaos, chaos suppression, and monitoring dynamical systems. As the result of these intensive studies, the framework of synchronized chaos in the form of identical oscillations has been well understood. However, taking into account the variety of forms of synchronized periodic oscillations, one understands that the regime of identical chaotic oscillations is just a particular form of synchronized chaos, and other, more complex forms of synchronized chaotic oscillations, can exist. Despite the fact that the first steps toward the understanding of such complicated cases of synchronized chaos were carried out by Afraimovich *et al.*[10] more than ten years ago, the general problem of synchronized chaos has still not been studied.

It is known that even for periodic oscillators, the theoretical analysis of synchronization in general is a very complicated problem. However, experimental analysis of the synchronized periodic oscillations is straightforward. Indeed, in order to observe the onset of synchronization in periodic oscillators, one needs to detect the formation of a stable limit cycle in the joint phase space of the coupled systems. This can easily be done by analysis of the Lissajous figures obtained by direct measurements from the oscillators. The experiment enables one to study the onset of synchronized periodic oscillations, despite the complexity of the forms of the limit cycles, and other complicating factors. These advantages of an experimental approach in studies of complicated forms of synchronization make it a very powerful tool for the exploration of synchronized chaos.

In this paper we employ the experimental results observed in coupled chaotic circuits to discuss the state of the art of chaos synchronization. In order to be more specific, we base the discussions upon the circuits generating smooth chaotic waveforms. In Sec. II we consider an example of well-known synchronized chaotic behavior, which is characterized by identical chaotic oscillations. Using this example we discuss the issues of the existence and stability of synchronized chaotic motions, and mechanisms that are responsible for the onset of chaos synchronization. In Sec. III we discuss experiments with directionally coupled circuits in which synchronization of chaos leads to richer behavior than the regime of identical chaotic oscillations. We also discuss an experimental approach that we use for studies of these forms of synchronization. In Sec. IV we present experimental results that indicate the possibility of synchronization of chaos with external periodic signals. The specific features of

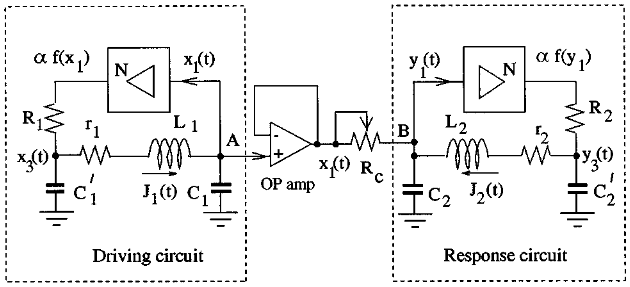

FIG. 1. The diagram of two dissipatively coupled chaotic circuits. The case of directional coupling is shown. In the case of mutual coupling the points A and B are connected through the resistor R_c without the OP amplifier.

chaos synchronization observed in relaxational oscillators are briefly discussed in Sec. V. Appendix A contains a brief discussion of mechanisms for synchronization of periodic oscillators. In Appendix B we give details on implementation of chaotic circuits used in our experiments.

II. SYNCHRONIZED CHAOS IN FORM OF IDENTICAL OSCILLATIONS

In this section we will consider a particular case of synchronized chaos where two coupled systems evolve identically in time. Synchronized chaotic oscillations of this form have been studied in detail in the vast literature on the subject. There are two categories of systems where this sort of behavior is usually investigated. The first category includes coupled systems that at zero coupling are identical to each other, and where each display chaotic behavior.[9–12] When the appropriate coupling is introduced, the systems demonstrate identical oscillations with the onset of synchronization. The second category, introduced by Pecora and Carroll,[13,14] consists of a driving system that exhibits chaotic behavior and of a response system. The latter replicates a portion or portions of the driving system. The variables that are not produced in the portion are taken from the driving system as the driving signals.

A. Synchronization with resistive coupling

Let us consider the dynamics of two identical chaotic circuits coupled to each other with a resistor; see Fig. 1. The details on the implementation and the individual dynamics of the circuit are briefly discussed in Appendix B. Usually, two types of the resistive coupling are considered. The first type is mutual coupling. This coupling is provided by connecting the similar points of the circuit through a resistor. The second type is directional coupling, where the same points are connected through the resistor and a unity-gain OP amplifier. The directional dissipative coupling can be considered as a proportional feedback control.

1. Synchronization manifold

The model describing the dynamics of two coupled identical circuits presented in Fig. 1 can be written in the form

$$\dot{x}_1 = x_2 + \epsilon_{21}(y_1 - x_1),$$
$$\dot{x}_2 = -x_1 - \delta x_2 + x_3, \qquad (1)$$
$$\dot{x}_3 = \gamma[\alpha f(x_1) - x_3] - \sigma x_2,$$
$$\dot{y}_1 = y_2 + \epsilon_{12}(x_1 - y_1),$$
$$\dot{y}_2 = -y_1 - \delta y_2 + y_3, \qquad (2)$$
$$\dot{y}_3 = \gamma[\alpha f(y_1) - y_3] - \sigma y_2.$$

The shape of nonlinearity $f(x)$ and the dependence of the parameters upon the physical values of the circuits elements are discussed in Appendix B. In the case of mutual coupling, $\epsilon_{21} = \epsilon_{12}$. In the case of directional coupling $\epsilon_{21} = 0$. The values of coupling parameters are proportional to the conductance of the resistor R_c,

$$\epsilon_{12} = \frac{1}{R_c}\sqrt{\frac{L}{C}}. \qquad (3)$$

It is easy to see that in the six-dimensional phase space of the systems (1), (2) there exists a three-dimensional integral manifold,

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = y_3. \qquad (4)$$

The phase space trajectories of the circuits that are located on the manifold given by (4), correspond to identical oscillations in both systems. One can easily see, from Eqs. (1) and (2), that when the systems behave identically, the behavior of the coupled system is the same as that exhibited by the individual systems without coupling. Therefore, if the parameters of the circuits are chosen in a region associated with chaotic behavior, and the manifold (4) is stable, then the circuits can demonstrate synchronized chaotic oscillations.

2. Dissipative coupling

Figure 2 shows the onset of synchronization in the experiment with directional coupling. Due to the use of the unity-gain OP amplifier in the coupling, the response circuit does not influence the behavior of the drive circuit, therefore the chaotic oscillations in the drive circuit do not depend upon the strength of the coupling; see Fig. 1. In the experiment, the inner parameters of the circuits were set to the values where both uncoupled circuits generate chaotic oscillations, which correspond to the attractors shown in Fig. 2(a). Due to the local instability of trajectories of these attractors, the oscillations in the uncoupled circuits ($R_c \to \infty$) are not correlated. The chaotic oscillations in the drive and in the response circuits can be considered as two identical, but independent, modes of oscillation. When the coupling is introduced, then the driving mode will tend to suppress the uncorrelated behavior of the initially excited mode of the response circuit. The mechanism of this suppression is the dissipation of energy in the coupling resistor. The difference between the voltages $[x_1(t)$ and $x_2(t)]$ induces a current though the resistor R_c. With a low value of the resistance R_c, a higher current is induced in the resistor. Therefore, for the same voltage difference $[x_1(t) - x_2(t)]$, the energy dissipation in the coupling will be larger when the resistance has a lower value. If the dissipation is sufficient for suppres-

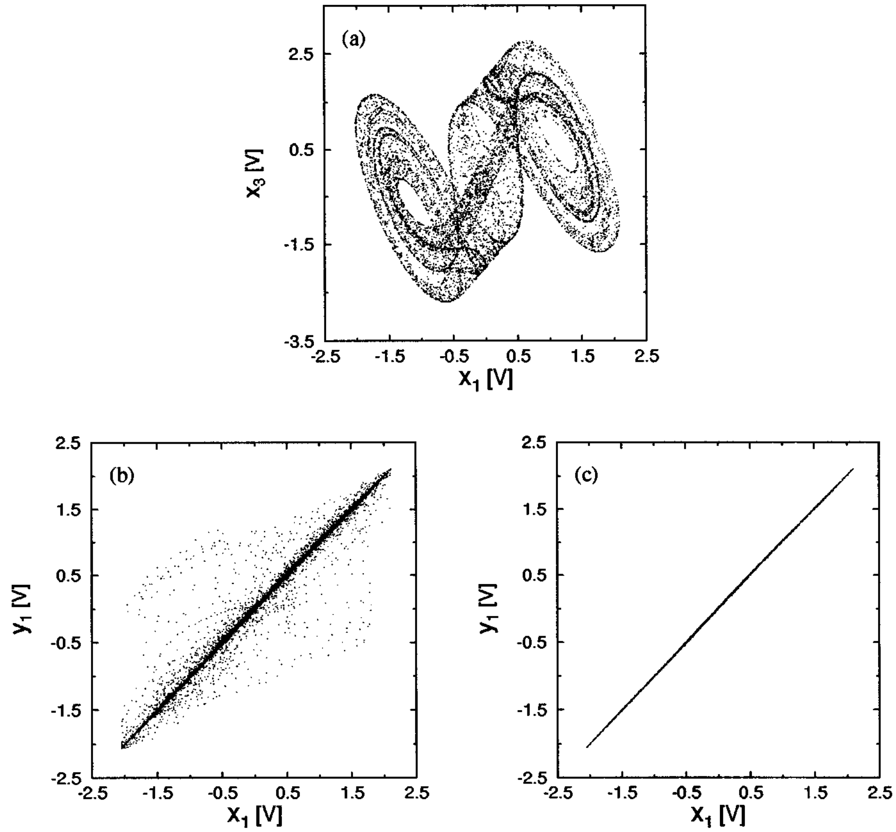

FIG. 2. (a) The projection of the chaotic attractor onto the plane (x_1, x_3). These are measurements from the driving circuit. The projection of the chaotic attractor on the plane $[x_1(t), y_1(t)]$ are plotted in (b) and (c). (b) Unsynchronized oscillations ($R_c = 1.0$ kΩ); and (c) synchronized chaotic oscillations ($R_c = 0.2$ kΩ). The parameter values of the drive and response circuits are $C_1' \approx C_2' \approx 220$ nF, $C_1 \approx C_2 \approx 332$ nF, $L_1 = L_2 = 144.9$ mH, $r_1 \approx r_2 \approx 348$ Ω and $R_1 = R_2 = 4.01$ kΩ, $\alpha_1 = \alpha_2 = 22.5$.

sion of the inner instabilities in the response circuit, then both circuits demonstrate identical chaotic oscillations. In this case, time evolution of all variables of the response circuit is identical to the time evolution of similar variables of the drive circuit. Figures 2(b) and 2(c) show the projections of the chaotic attractors measured in the circuits, with the values of coupling taken below and above the threshold for the onset of synchronization.

3. Chaos synchronization and stability

In order to define the threshold for chaos synchronization, one needs to study the stability of synchronized trajectories as a function of the coupling parameter. Since in our case the synchronized oscillations are chaotic, these trajectories are always unstable. However, in the analysis of synchronized chaotic motions one has to distinguish between the instability for perturbations tangent to the manifold (4) and transverse to it. The regime of identical chaotic oscillations is stable when the synchronized trajectories are stable for the perturbations in the transverse direction to the synchronization manifold (4).

Two most frequently used criteria for stability of synchronized chaotic motions are the Lyapunov function criterion (see, for example, Refs. 15–18) and the analysis of transversal (conditional) Lyapunov exponents calculated from the linearized equations for the perturbations transversal to the synchronization manifold (see, for instance, Refs. 9, 13, 19, and 20). The criterion based on the analysis of Lyapunov functions for the vector field of perturbations transversal to the manifold enables one, in some cases, to prove that all trajectories in the phase space of the coupled systems are attracted by the manifold of synchronized motions. Despite the fact that this criterion guarantees the onset of synchronization, it is not a general method since there is no procedure for constructing the Lyapunov function for an arbitrary system. In many practical cases, Lyapunov functions cannot be found, even for systems that possess a stable manifold of synchronized motions for a broad range of parameters of coupled systems, and of the coupling itself.

In contrast with Lyapunov functions, the analysis of transversal Lyapunov exponents is quite straightforward and can be easily employed, even for rather complicated systems. However, it has been pointed out[21–25] that, in practice, the negativeness of Lyapunov exponents does not always guarantee the onset of synchronized motions. Even in the cases where all transversal Lyapunov exponents are negative, there may exist atypical trajectories in the immediate vicinity of the manifold of synchronized motion that depart from the manifold exponentially fast. The appearance of such trajectories is responsible for destabilization of the synchronized chaotic motions. The synchronized chaotic motions become unstable, in the sense that vanishing noise in the coupling and/or internal noise in the coupled systems and/or small mismatch between the parameters of the systems lead to the loss of the synchronization and to the bubbling behavior considered in Refs. 22, 24, and 25.

The applications of the Lyapunov function criteria and the analysis of transversal Lyapunov exponents for the stability of the synchronized motions in the model of coupled circuits (1) and (2) can be found in Ref. 16 It has been also shown in the same paper that for strong enough coupling the synchronized chaos in these circuits is robust. The robustness implies that small perturbations of the parameters of the systems lead to small deviations from the identical oscillations. The stability and robustness of synchronized chaotic motions is easily determined in experimental studies. Indeed, due to the natural noise and small parameter mismatch that are always present in the experiment, the regime of practically identical oscillations is observed only in the case where the synchronized solutions are stable against transversal perturbations and robust. Therefore, the experimentally obtained Fig. 2(c) shows the chaotic attractor located on the *stable* manifold of synchronized motions.

4. Mechanism for chaos synchronization

The onset of the synchronization between chaotic oscillators of smooth waveforms differs from the case of periodic oscillation. Unlike the case of synchronization between identical periodic oscillators, the synchronization between identical chaotic oscillators is, generally, characterized by a certain threshold value of the coupling that is required for the onset of the synchronization. The reason for the existence of this threshold value is the local instability of the chaotic trajectories. In order to achieve synchronization in directionally coupled circuits, the instability in the response circuit has to be suppressed. In other words, the amplitude of the mode of the oscillations induced in the response circuit by the driving has to be sufficient to compete and suppress the ''independent'' mode in the response circuit. In the case of mutual coupling, the dissipation induced by the coupling causes the onset of the energetically preferred regime of mutual behavior. Depending upon the parameters of the coupling, and the nonlinearity and dissipation in the circuits, the preferred regime of oscillations can be the mode of identical chaotic oscillations, nonidentical stable periodic and quasiperiodic oscillations, and, even, fixed states. Discussion of the bifurcations in the mutually coupled circuits (1) and (2) that provide the alternations between of these regimes can be found elsewhere.[16]

It is clear that in some cases the dissipation of energy in the coupling between chaotic systems can fail to suppress transversal instability. For example, there are system in which some of the variables do not sufficiently influence the instability, which is provided by the dynamics associated with the other variables of the system. In these cases even infinitely strong dissipation in the coupling that uses the first group of variables may seem incapable of suppressing the transversal instability in coupled systems. Moreover, there are known examples where the synchronization of chaos is observed only within an interval of moderate values of the coupling parameter; see, for example, Ref. 26. This interval of the parameter values is restricted both from below and from above. The role of coupling in this case is not only to provide an extra dissipation, but also to change the inner properties of the synchronized system in order to provide a better interaction between different portions of the system.

B. Stabilization of unstable orbits by means of synchronization

Considering the case of the directionally coupled circuits one can notice that the synchronized motions on the manifold (4) correspond not only to identical chaotic motions but also to identical unstable periodic orbits embedded in the synchronized chaotic attractor. Since the manifold is stable, the instability of these periodic orbits is caused only by the instability of the corresponding orbits in the phase space of the driving circuit. Therefore, if one takes a waveform $x_{1p}(t)$ that corresponds to a unstable periodic orbit embedded in the chaotic attractor of the driving circuit, and drives the response circuit with the periodic signal $x_{1p}(t)$, then, due to the stability of the synchronization manifold (4), the response and driving circuit will operate in the stable regime of identical periodic oscillations.

It is important that when the systems operate in the regime of identical oscillations the term of the system (2), which describes the coupling, becomes equal to zero. It means that when the response circuit reaches the desired regime of oscillations, the current through the coupling resistor

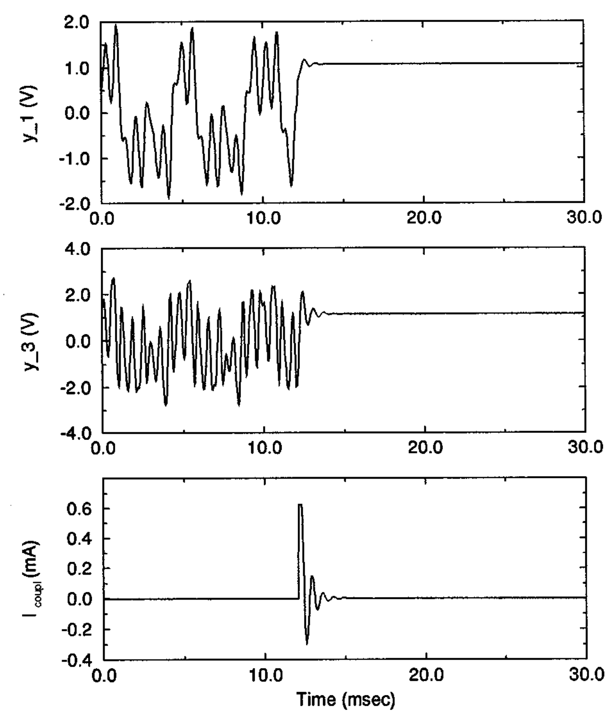

FIG. 3. An example of time series $y_1(t)$, $y_3(t)$, and $I_{\rm coupl}(t)$ during the transition to the fixed point O_R after applying the driving at time $t_{\rm start}$=12.2 ms.

R_c becomes equal to zero. In practice, due to noise and small deviation of the parameters of the circuits, the system always requires some current through the coupling resistor, but this current can be very small. Figure 3 shows an example of transition from chaotic motions to a stabilized fixed point after the appropriate driving is applied. In this case, the unstable fixed point does not belong to the chaotic attractor. However, since the fixed point is a solution of uncoupled response circuit, the current I_{coupl} through the coupling resistor becomes exponentially small after the system approaches the fixed point. Details of this experiment can be found elsewhere.[27]

The technique of chaos suppression by driving the system with matched periodic signals was studied by Pyragas.[26] Due to simplicity of implementation and transparent theoretical background, this technique is easily applicable in different experiments; see, for example, Refs. 27 and 28.

III. GENERALIZED CASES OF SYNCHRONIZED CHAOS IN DRIVE-RESPONSE SYSTEMS

In the previous section we considered synchronized chaotic oscillations in the form of identical oscillations. In many recent papers, the onset of identical oscillations is used as a definition of synchronization. It is clear that such an interpretation of synchronized chaos significantly narrows the meaning of this phenomena. Indeed, taking into account the variety of different forms of synchronized behavior in coupled periodic oscillators, one can expect that synchronization of chaos in the form of identical oscillations is only a particular case of synchronized chaos. More general cases of the synchronization include the synchronization between chaotic systems with different parameters. In such cases the synchronized chaotic trajectories are located on a stable synchronization manifold, which is no longer a hyperplane (4), but a more complex geometrical object. In order to distinguish these ''nontrivial'' cases from the cases of identical chaotic oscillations, which are usually used in the literature to define synchronized chaos, we call such synchronization *generalized* synchronization of chaos.[29]

The main difficulty of theoretical analysis of the synchronization between different chaotic system is to define a set of common features of the dynamical behavior that will enable one to clearly distinguish the synchronized chaotic motions from unsynchronized ones. Indeed, depending on the coupling parameters, two chaotic system can, generally, demonstrate different forms of chaotic, quasiperiodic, and periodic oscillations. It is clear that the periodic oscillations can be considered as the onset of periodic synchronization. However, since the individual behavior of the systems is chaotic, then these systems can demonstrate synchronized chaotic oscillations. In the general case, the synchronized chaotic oscillations are different from the chaotic oscillations generated by the uncoupled systems. Therefore, the similarity between the synchronized chaotic attractor and the chaotic attractors of uncoupled systems cannot be considered as a requirement for the synchronized chaos. The first mathematical definition of the synchronized chaos, which includes the case of nonidentical oscillations, has given in the paper by Afraimovich, Verichev, and Rabinovich.[10] This definition is based on the idea of a homeomorphous transformation, which is required for linking the projections of the synchronized chaotic trajectories onto phase subspaces of the coupled systems. However, this mathematical definition contains a number of conditions that cannot be shown to be satisfied in real experiments with nonidentical chaotic systems. Moreover, we believe that in some cases the requirement of the homeomorphism between the projections excessively restricts the meaning of synchronized motions.

In this section we will discuss simplified cases of nonidentical synchronized chaotic oscillations that are observed in directionally coupled circuits with different parameter values. In these cases the dynamics of the driving systems does not depend upon the behavior of the response system, therefore the synchronized oscillations should be related to the attractor in the driving system.

A. The auxiliary system method for detecting synchronized chaos

1. Synchronization and predictability

In order to define a common feature that allows one to distinguish synchronized behavior, we refer to general cases of synchronized periodic oscillations in drive-response systems. Despite the complexity of the synchronized periodic oscillations, they always satisfy the property that is the ability to predict the states of the response system from observations of the drive system. Due to the stability of the synchronized motions, the response behavior is not sensible to small perturbations of the initial conditions of the systems.

It was shown in Ref. 29 that the synchronization between chaotic driving and response systems can also be detected through the analysis of such predictability. The ability to predict the current state of the response system from the chaotic data measured from the driving system can be used as a definition of synchronized chaos in a generalized sense. The predictability indicates the existence and stability of a chaotic attractor located in the synchronization manifold. The behavior of the system on the manifold is controlled only by the motions in the phase space of the driving system. Note that, in general, the synchronization manifold can have a very complicated shape and cannot be detected from simple observations of its projections, even when the manifold is stable. When the systems lose synchronization, the driving system does not provide complete control of the behavior of the response system and small perturbations in the response system will grow.

2. The auxiliary system

In the experiments we use a replica of the response system as a predicting device. This auxiliary system is driven in the same way as the response system and has no connection with the response system. Therefore, the regime in which the auxiliary system is synchronized with the driving system is stable whenever the response system is synchronized. Note

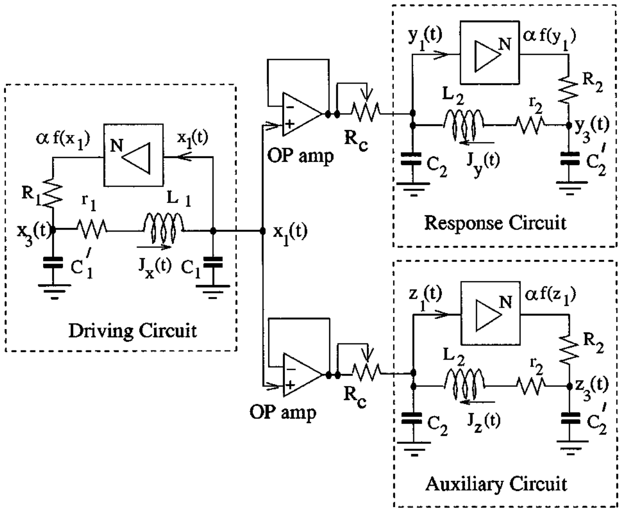

FIG. 4. The circuit diagram of the experiment with driving, response, and auxiliary nonlinear electrical circuits. The parameters values of the drive circuit are set to be $C_1' = 230$ nF, $C_1 \approx 337$ nF, $L_1 \approx 140$ mH, $r_1 \approx 334\ \Omega$, and $R_1 \approx 4.21$ kΩ. The parameter values of the response and auxiliary circuits are $C_2' \approx 225$ nF, $C_2 \approx 342$ nF, $L_2 \approx 145$ mH, $r_2 \approx 348\ \Omega$, and $R_2 \approx 4.97$ kΩ.

that both response and auxiliary systems may be synchronized to the driving system in the generalized sense. In this regime due to the identity between the parameters of the response and auxiliary systems they demonstrate identical oscillations. Therefore, the auxiliary system can be considered as the ideal predictor that is able to indicate the current state of the response system by processing the driving signal. At the same time, these identical chaotic oscillations can be easily detected. The onset of identical chaotic oscillations in response and auxiliary systems in the presence of small noise and parameter mismatch (which are unavoidable in the experiments) guarantees the stability of the synchronization manifold, and therefore guarantees synchronization between the driving and response systems.[30]

The auxiliary system approach can also be useful for theoretical analysis of the generalized synchronization of chaos. This approach enables one to reduce the study of the transversal stability of very complicated synchronized motions, to the analysis of transversal stability for identical chaotic oscillations. However, we have to emphasize that the issues of the robustness, which are briefly discussed in Sec. II, are crucial for this analysis.

B. Generalized synchronization in systems with dissipative coupling

To demonstrate the generalized synchronization of chaos in an experiment with electronic circuits, we built two almost identical electronic circuits that were driven by a chaotic signal from a third circuit. The circuit diagram of the experiment is shown in Fig. 4. The chaotic signal generated by the drive circuit was applied to both the response and the auxiliary circuits through the resistors R_c. The strength of the coupling was controlled by the values of R_c in each circuit. This was adjusted to have the same value for both circuits. In the experiment we tuned the parameters of the drive circuit to correspond to the regime of chaotic oscillations. This attractor is shown in Fig. 5(a). The parameters of the response and the auxiliary circuits were tuned to values, which, without coupling, namely, $R_c \rightarrow \infty$, lead these circuits to generate chaotic oscillations corresponding to the attractor shown in Fig. 5(b).

Synchronization of the chaotic oscillations was observed for values of the coupling with $R_c < 630\Omega$. It was easily detected with the analysis of the projections of the synchronization manifold onto the planes (y_1, z_1) and (y_3, z_3). For the synchronized oscillations, the trajectory is projected into the diagonals $y_1 = z_1$ and $y_3 = z_3$ on these planes. These identities guarantee the identity of the currents $J_y(t) = J_z(t)$, which one can see in Fig. 4. Synchronized behavior, observed with $R_c = 604\Omega$, is shown in Fig. 6. The synchronized chaotic attractor measured in the response circuit is presented in Fig. 6(a). The fact of the synchronization is confirmed by the stability of the ''diagonal manifold'' in the state space of *response+auxiliary* system [see Fig. 6(b)], from which the stability of the manifold of synchronized motions in the phase space of *drive+response* system follows. Looking at the projections of the synchronized chaotic attractors onto the plane of the variables (x_1, y_1) and (x_3, y_3), it becomes

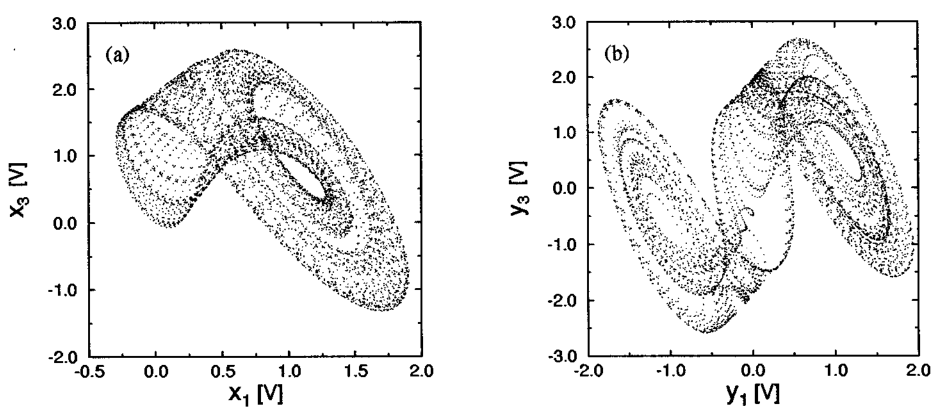

FIG. 5. The experimentally measured chaotic attractors of the uncoupled drive (a) and response (b) circuits. The parameters in the nonlinear converters N in the drive is $\alpha \approx 22.85$, and in the response it is $\alpha \approx 24.62$. The attractors in the two systems are not the same, as the systems are different.

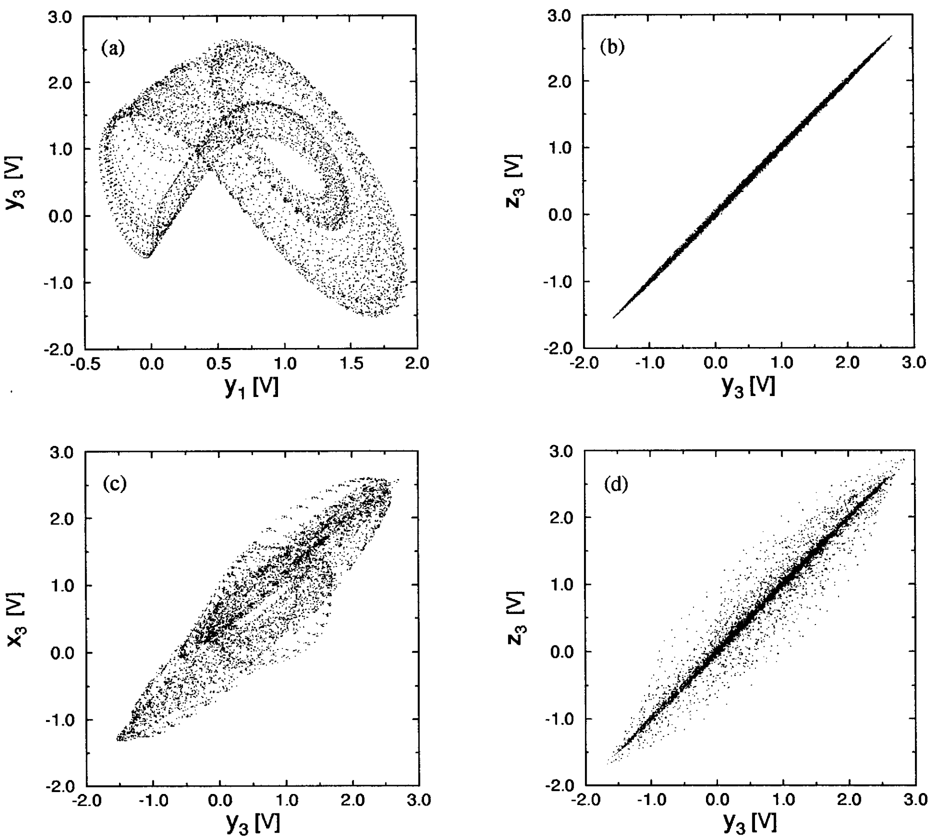

FIG. 6. The projections of synchronized (a)–(c) and unsynchronized (d) chaotic motions measured with $R_c = 604\ \Omega$ and $R_c = 731\ \Omega$, respectively. Sampling rate 20 μs. (a) The synchronized chaotic attractor measured in the response circuit. (b) The projection of the chaotic motions onto the plane of the variables (y_3, z_3) measured from response and auxiliary circuits. (c) The projection of the chaotic motions onto the plane of the variables (y_3, x_3) measured from response and driving circuits. (d) The projection of the chaotic motions onto the plane of the variables (y_3, z_3) measured from response and auxiliary circuits.

clear that oscillations in the driving and response circuits are not identical, see Fig. 6(c). Therefore, these circuits are synchronized in the generalized sense.

Unsynchronized chaotic oscillations, measured with $R_c = 731\Omega$, are shown in Fig. 6(d). The projection of these chaotic oscillations onto the plane (y_3, z_3) clearly indicates that these oscillations are not synchronized. Indeed, one can see that trajectory frequently leaves the close vicinity of the synchronization manifold. This means that the manifold is transversely unstable.

C. Synchronized chaos in systems with different frequencies

In this section we present an example of synchronized chaos in which the coupled systems exhibit oscillations with distinct characteristic frequencies. The circuit diagram of the experiment is shown in Fig. 7. The synchronizing signal $x_1(t)$ from the driving circuit is applied to the response circuit, where it is mixed with the signal $\alpha f[y_1(t)]$.

To explore the generalized synchronization of chaos with frequency ratio 1:2, in the experiment we set the parameters of the circuits to the values shown in the Fig. 7 caption. Values of the control parameters, α, for the nonlinear converters in the driving and response circuits were $\alpha_1 = 22.86$ and $\alpha_2 = 14.0$, respectively. As the coupling parameter we

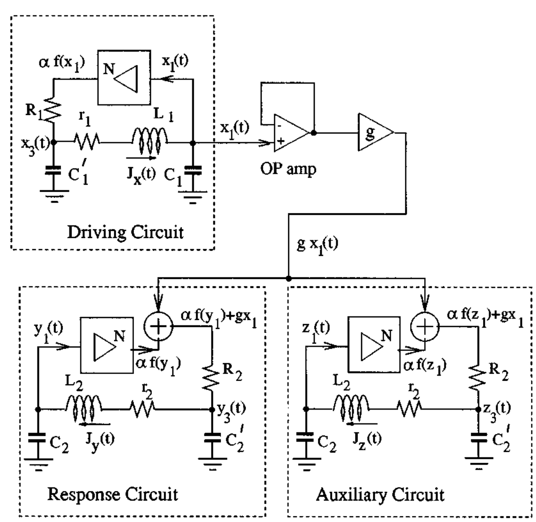

FIG. 7. The diagram of the experimental setup. The parameters values of the drive circuit are set to be $C_1' = 221$ nF, $C_1 \approx 336$ nF, $L_1 \approx 144$ mH, $r_1 \approx 358\ \Omega$, and $R_1 \approx 4.69$ kΩ. The parameter values of the response and auxiliary circuits are $C_2' \approx 105$ nF, $C_2 \approx 160$ nF, $L_2 \approx 75$ mH, $r_2 \approx 161\ \Omega$, and $R_2 \approx 4.32$ kΩ.

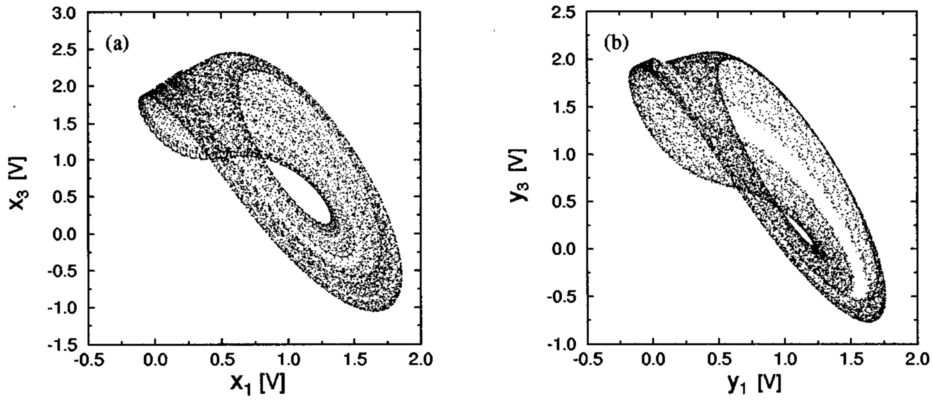

FIG. 8. The projections of the chaotic attractors measured in uncoupled circuits with sampling rate 20 μs. (a) (x_1, x_3) projection of the attractor in the driving circuit. (b) (y_1, y_3) projection of the attractor in the response circuit, $g=0$.

use the amplification, g, of the driving signal, $x_1(t)$, in the voltage mixtures $\alpha f[y_1(t)]+gx_1(t)$ and $\alpha f[z_1(t)] + gx_1(t)$; see Fig. 7.

1. Synchronized chaos

The chaotic attractors measured from the driving circuit and the response circuit without driving $(g=0)$ are shown in Figs. 8(a) and 8(b), respectively. Despite the similarity of the shapes of the chaotic attractors the oscillations in the circuits are generated with different characteristic frequencies. The response circuit oscillates twice as fast as the driving circuit. The attractor in the auxiliary circuit looks exactly the same as the attractor in the response circuit shown in Fig. 8(b), but the oscillations between the auxiliary and response circuits without driving are not correlated.

Let us consider the synchronized chaotic oscillations that were observed with $g=2.0$. Different projections of the syn-

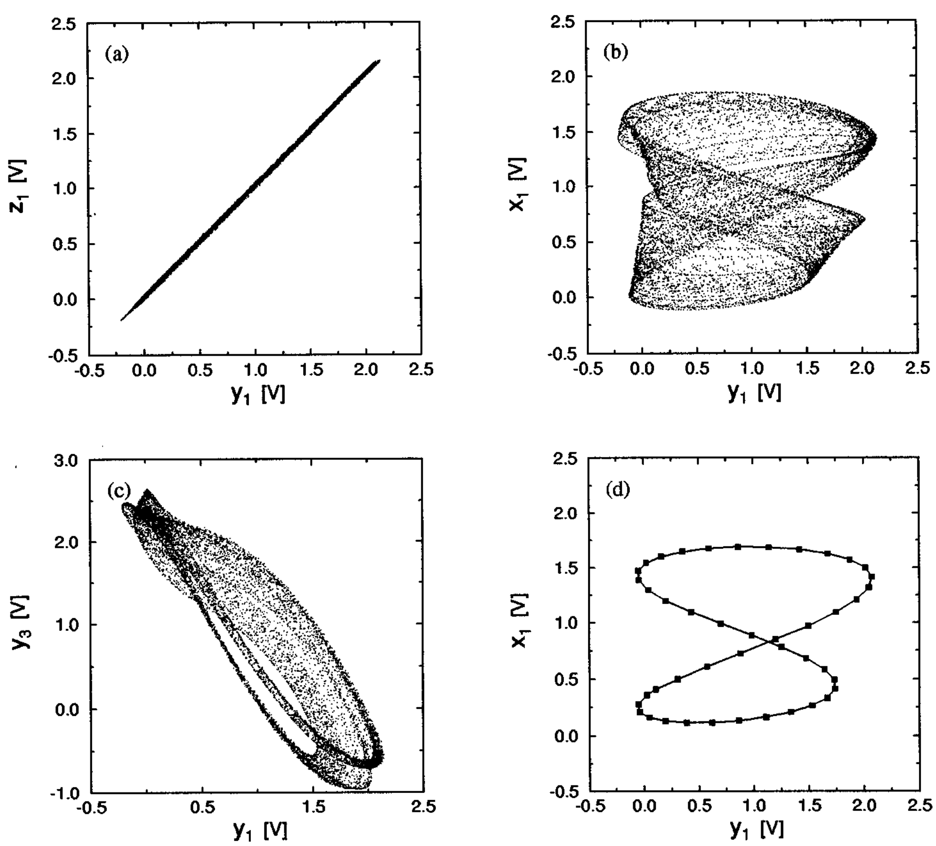

FIG. 9. The projections of synchronized chaotic motions measured with $g=2.0$. Sampling rate 20 μs. (a) The projection of the chaotic motions onto the plane of the variables (y_1, z_1) measured from response and auxiliary circuits. (b) The projection of the chaotic motions onto the plane of the variables (y_1, x_1) measured from response and driving circuits. (c) The synchronized chaotic attractor measured in the response circuit. (d) The Lissajous figure of the unstable period-1 orbit retrieved from the chaotic data.

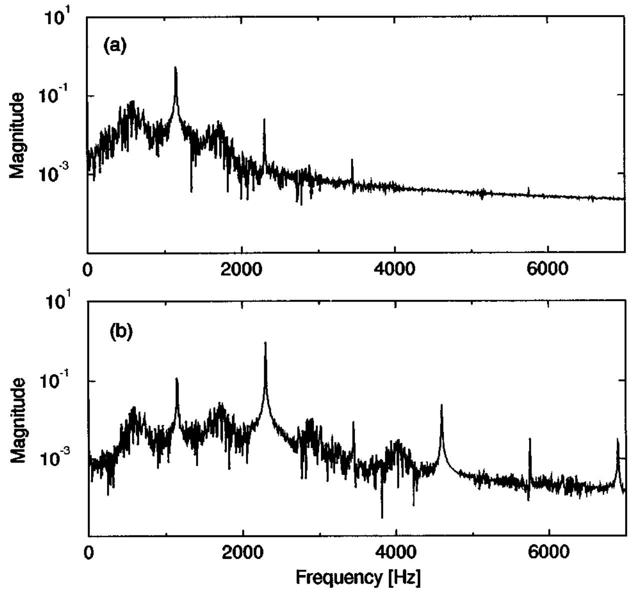

FIG. 10. (a) The spectrum of the driving signal, $x_1(t)$. (b) The spectrum of the signal, $y_1(t)$, measured from the synchronized response circuit.

chronized chaotic attractor are shown in Fig. 9. The fact that the driving and response circuits are synchronized follows from the identity of chaotic oscillations measured from the auxiliary and response circuits; see Fig. 9(a). For the same oscillations, the projection of the synchronized motions onto the plane of variables of the drive and response circuits has a rather complicated form, which is shown in Fig. 9(b). The synchronized attractor measured in the response circuit differs from the attractor of the driving circuit; compare Fig. 8(a) and Fig. 9(c).

The frequency spectra of the signals $x_1(t)$ and $y_1(t)$ are shown in Fig. 10. From these figures one can easily see that the synchronized oscillations in drive and response circuits have different characteristic frequencies.

2. Synchronization of unstable periodic orbits

We found that the pairs of unstable periodic orbits embedded in the chaotic attractors of the driving and response circuits are phase locked with the same frequency ratio $\omega_d/\omega_r = \frac{1}{2}$, where ω_d and ω_r are the main frequencies of the orbits in driving and response circuits. Figure 9(d) shows the Lissajous figure for the simplest unstable periodic orbit (period-1) retrieved from the experimental chaotic data. One can see that moving along the period-1 orbit the response system makes two rotations while the driving system makes only one. Similar Lissajous figures were found for other unstable periodic orbits embedded in the synchronized chaotic attractor. From this analysis it is reasonable to assume that the transition to synchronized chaos is accompanied by phase locking of pairs of unstable limit cycles existing in the chaotic attractors of uncoupled driving and response systems. Of course, we cannot demonstrate experimentally that *all* pairs of saddle periodic orbits in driving and response circuits are phase locked when the chaotic oscillations in the circuits are synchronized. However, this seems quite plausible and this phase locking can be considered as the mechanism of the onset of synchronized chaos. Since in our experiment the synchronization of periodic orbits is characterized by the frequencies ratio 1:2 this ratio can be considered as a characteristic of the synchronized chaotic oscillations.

D. Regularization by chaotic driving

In this section we briefly discuss phenomenon of regularization by chaotic driving, which is observed in directionally coupled chaotic systems synchronized with frequency ratio 2:1. The essence of this phenomena is that the behavior of the chaotic system driven by another chaotic system becomes more regular than it was in either of these two systems when they were uncoupled. In order to achieve chaos synchronization with frequency ratio 2:1 we use the same experimental setup as shown in Fig. 7, but the parameters of the drive, response and auxiliary systems are set to be $C_1' = 108$ nF, $C_1 \approx 160$ nF, $L_1 \approx 75$ mH, $r_1 \approx 162\ \Omega$, and $R_1 \approx 4.32$ kΩ. The parameters values of the response and auxiliary circuits are $C_2' \approx 224$ nF, $C_2 \approx 340$ nF, $L_2 \approx 145$ mH, $r_2 \approx 357\ \Omega$, and $R_2 \approx 4.69$ kΩ. Values of the control parameters, α, for the nonlinear converters in driving and response circuits are $\alpha_1 \approx 14$ and $\alpha_2 \approx 21.4$. Chaotic oscilla-

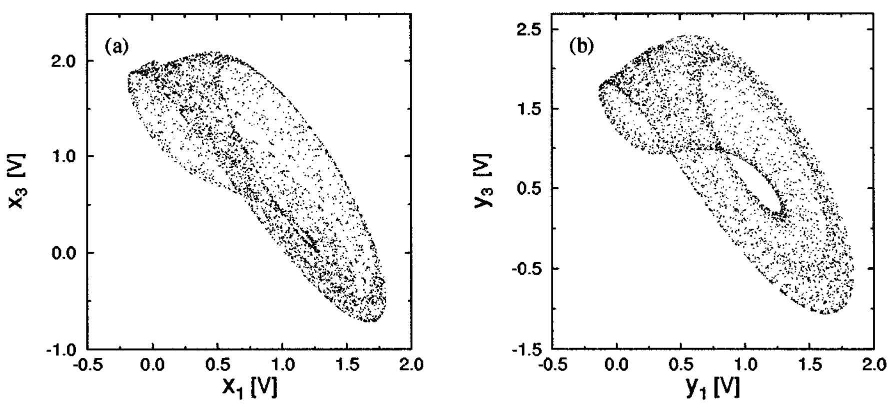

FIG. 11. The projections of the chaotic attractors measured in uncoupled circuits with sampling rate 20 μs. (a) (x_1, x_3) projection of the attractor in the driving circuit. (b) (y_1, y_3) projection of the attractor in the response circuit, $g = 0$.

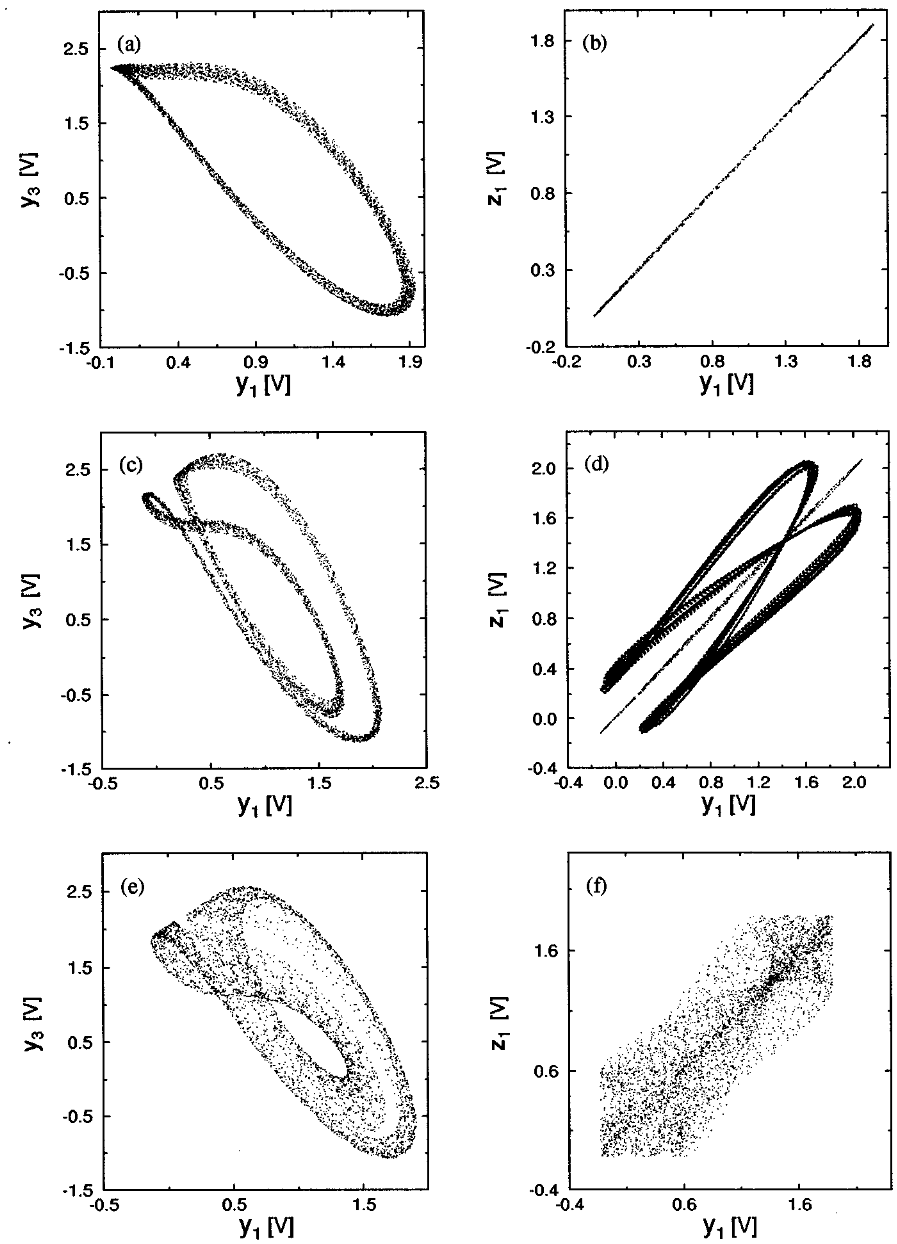

FIG. 12. Projections of chaotic attractor onto the plane of phase variables (y_1, y_3) in the response circuit (left) and onto plane of variables (y_1, z_1) of auxiliary and response circuits (right). The parameters of the coupling are (a) and (b) $g=2.0$, (c) and (d) $g=1.6$, (e) and (f) $g=0.5$.

tions in uncoupled circuits $(g=0)$ correspond to attractors shown in Fig. 11. In this case the driving circuit oscillates twice as fast as the response circuit.

1. Synchronization and regularization

When the coupling parameter is set to the value $g=2.0$, we observe synchronized chaotic oscillations with frequency ratio 2:1. The fact of the onset of synchronization was detected via analysis of oscillations in the auxiliary and response circuits. The projections of the synchronized chaotic attractor are presented in Figs. 12(a) and 12(b). Comparing this synchronized chaotic attractor [Fig. 12(a)] with the attractors of the uncoupled circuits (Fig. 11), one can see that synchronized chaotic oscillations are more regular than in both the uncoupled response circuit and the driving circuit. The fact of the regularization is also confirmed by analysis of spectra of chaotic signals; see Fig. 13.

Studying this regime of synchronized oscillations, we found that regularization is typical within the synchronization zone with frequency ratio 2:1. A naive explanation for chaos regularization observed in the experiment can be the following. The chaotic driving signal $x_1(t)$ contains dc and periodic components; see Fig. 11(a). These components can be considered as additional parameters of the response circuit. The values of these parameters are proportional to the value of coupling parameter g. When the coupling is strong enough these parameters change the dynamics of the response circuit toward a regime of stable periodic motion. If

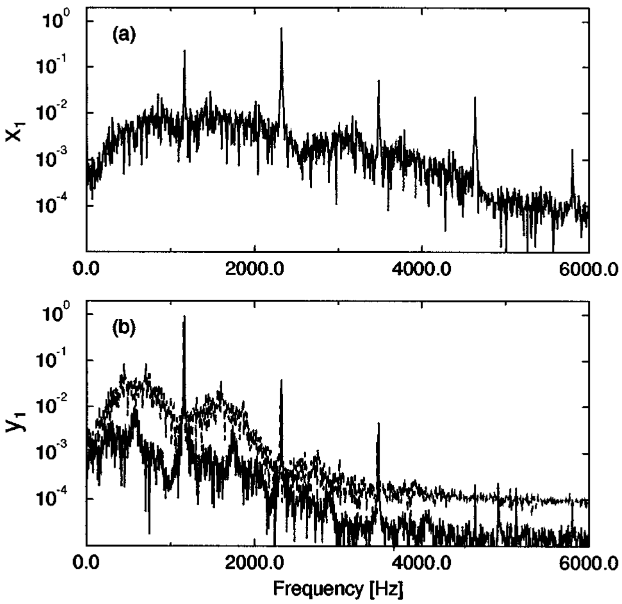

FIG. 13. (a) The spectrum of the driving signal, $x_1(t)$. (b) The spectrum of the signal, $y_1(t)$, measured from the uncoupled response circuit (dashed line) and the synchronized response circuit with $g=2.0$ (solid line).

this periodic motions is a ''strong'' attractor, then irregular components of the driving signal will be substantially suppressed in the response system. However, despite the fact that such changes in the response circuit are responsible for the onset of synchronization, this naive explanation cannot account for regularization by chaos observed in our experiment. First, the amplitude of chaotic component of the driving signal is also proportional to the value of coupling parameter. Second, comparing the values of dc components and the amplitudes of periodic components in the driving signals of this experiment [see Fig. 11(a)] and the experiment of synchronization with the frequency ratio 1:2 [see Fig. 8(a)] one can see that these parameters of the driving signals are almost the same in both cases. However, in the case of chaos synchronization with frequency ratio 1:2, the phenomena of regularization by chaos is not observed. Therefore, these experiments indicate that relation between characteristic frequencies of driving and response systems is crucial for such regularization.

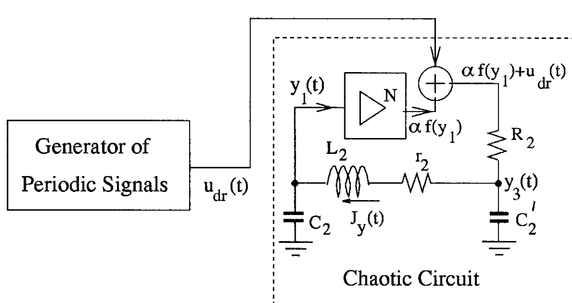

FIG. 14. The diagram of the experimental setup.

Due to the low-dimensionality of the chaotic attractor in the phase space of the driving system, the attractor can be considered as a closed band of the trajectories flow. During one rotation along the attractor this band increases its width, because of local instability of the trajectories, and then, the band folds to fit the initial width. Since the response circuit is synchronized with the frequency ratio 2:1, the band of the chaotic attractor in the response system is subject to double folding. As the result of the double folding the trajectories of the attractor in the response system are more twisted than it is in the driving system. We believe that regularization in the synchronized response system is associated with this multiple folding induced by the topology of the fast driving attractor. However, in order to support this statement more detailed numerical studies have to be done.

2. Auxiliary system and multistability

When the value of coupling parameter g decreases, we observe ''period doubling bifurcations'' associated with synchronized *chaotic* motions; see Figs. 12(a), 12(c), and 12(e). After the first period doubling bifurcation the response circuit has two regimes of synchronized motions. In the phase space of the response system both regimes correspond to the same attractor [see Fig. 12(c)]. The only difference between these regimes is the ''phase'' of the synchronized oscillations. These regimes of oscillations can be distinguished by means of an auxiliary system. In the phase space of response-auxiliary systems, these synchronized oscillations correspond to different attractors. Figure 12(d) shows the projections of these synchronized attractors onto the plane of variables of the response and auxiliary circuits. ''In-phase'' motions (shown by dots) correspond to the trajectories located on the diagonal; see Fig. 12(d). The trajectories of ''out-of-phase'' oscillations are shown by a dashed line. This example illustrates that the onset of identical oscillations in auxiliary and response systems is sufficient, but not necessary condition for chaos synchronization between drive and response systems.

The transition from ''period-1'' [Fig. 12(a)] to ''period-2'' [Fig. 12(c)] motions occurs not at a bifurcation point but within a certain interval of the coupling parameter values where the intermittent behavior is observed. This intermittency is characterized by switching the ''phases'' of synchronized ''period-2'' motions. From the viewpoint of predictability these intermittent chaotic oscillations are not synchronized. Outside this interval of the parameter values, the trajectories flows of the ''period-2'' attractor are well separated, and the behavior of the response circuit can again be predicted.

In the experiment we clearly detect two consequent period doubling bifurcations from the chaotic motions. Every such bifurcation doubles number of coexisting synchronized chaotic attractors in the phase space of response-auxiliary systems. After destabilization of ''period-4'' motions the response system forms chaotic attractor presented in Fig. 12(e). This attractor is characterized by the intermittency that is caused by ''phase'' switching within the synchronized

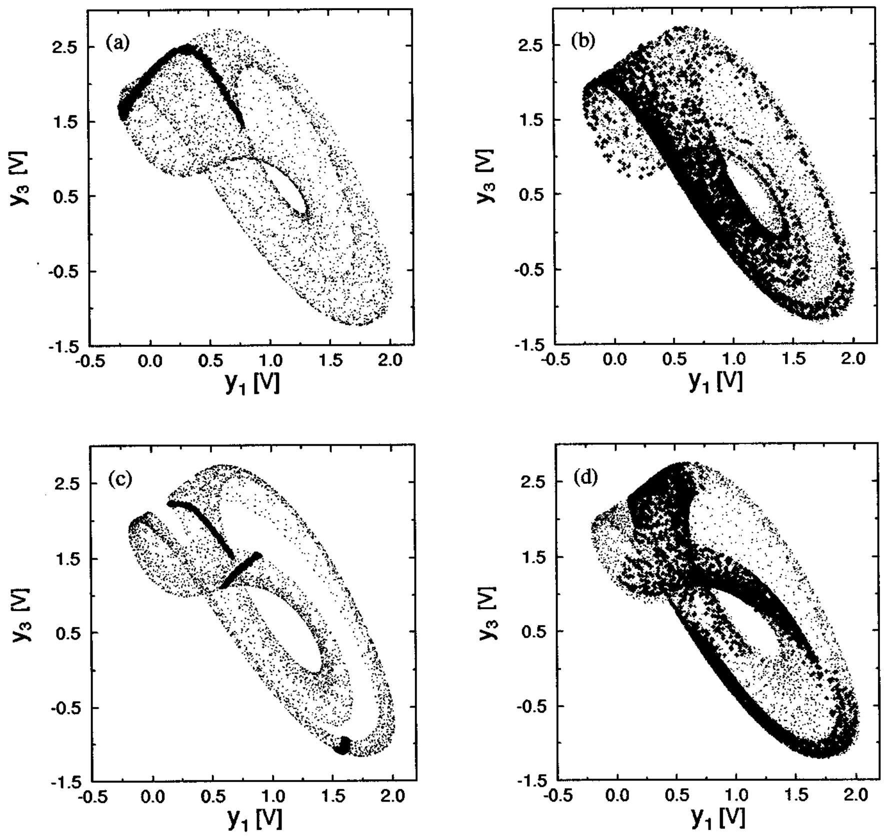

FIG. 15. The images of periodically synchronized and unsynchronized chaotic oscillations. Synchronized attractors are shown in (a) and (c). Unsynchronized attractors are shown in (b) and (d). Intersections of the trajectories with the Poincaré cross sections are shown by pluses (bright dots) against the background of the attractor projections, which are plotted by small dots. The parameters of the external driving are (a) $A=304$ mV, $\Phi=1.155$ kHz; (b) $A=304$ mV, $\Phi=1.169$ kHz; (c) $A=304$ mV, $\Phi=1.727$ kHz; (d) $A=304$ mV, $\Phi=1.723$ kHz.

chaotic set. Due to this switching the behavior of the response systems cannot be predicted; see Fig. 12(f).

IV. SYNCHRONIZATION OF CHAOS BY PERIODIC SIGNALS

As was indicated in the previous section, the onset of synchronized chaos is accompanied by phase locking of pairs of periodic orbits embedded in the chaotic attractors of driving and response systems. This phase locking between the periodic orbits can be considered as a possible bifurcation scenario for the transition from nonsynchronized chaotic oscillations to synchronized chaos. The phase locking of saddle periodic orbits can also be used to ''synchronize'' the chaotic oscillations to an external periodic signal. This ''synchronized'' chaotic behavior can be achieved if the external periodic signal can provide phase locking for the periodic orbits embedded in the chaotic attractor of autonomous system.

In order to demonstrate this type of synchronized behavior we will consider an experiment where the response circuit is driven by a periodic signal $u_{\mathrm{dr}}(t)=A\ \sin(2\pi\Phi t)$. The circuit diagram of the experiment is shown in Fig. 14. In the experiment the signal $u_{\mathrm{dr}}(t)$ is added to the signal $\alpha f(y)$, which is produced by the nonlinear converter N. The parameters of the response circuit are selected to have the same values as the parameters of the driving circuit shown in Fig. 7. As a result, the uncoupled response circuit (with $A=0$) generates the chaotic oscillations that correspond to the attractor shown in Fig. 8(a).

A. Phase locked chaotic attractors

We consider two examples of ''synchronized'' chaotic oscillations. In the first example the oscillations are synchronized with frequency ratio 1:1. In the second, the frequency ratio is 2:3. The chaotic attractors measured inside and outside the synchronization zone are shown in Fig. 15. In order to study the phase locking, the experimentally measured attractors were examined on the cross sections conditioned by the period of the external driving. In the plots of the attractors, the intersections of the trajectory with such Poincaré cross sections are marked by small pluses. In order to see where these intersections are located on the attractor, we plot these stroboscopic observations against the background of the projection of the attractor shown by small dots. Comparing the attractor in Fig. 15(a) with the attractor in Fig. 15(b), one can see that the chaotic oscillations of the first attractor are phase locked with the external periodic signal. Indeed, all intersections of the trajectory of the phase locked attractor are located in a narrow domain on the top of the attractor [see Fig. 15(a)], and the attractor itself looks very similar to

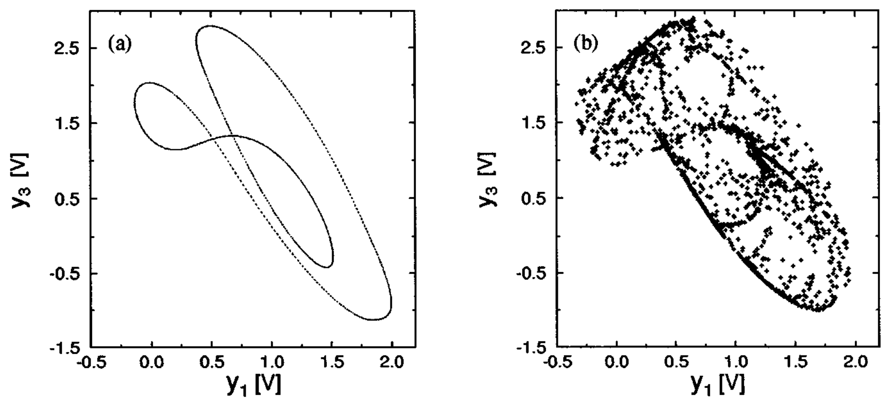

FIG. 16. (a) Projection of a stable periodic motion appeared inside the synchronization zone. $A=1.1$ V, $\Phi=1.723$ kHz. (b) The Poincaré cross section of the unsynchronized chaotic attractor appeared after the periodic motion becomes unstable. $A=1.52$ V, $\Phi=1.723$ kHz.

the chaotic attractor observed in the unperturbed system [compare Figs. 8(a) and 15(a)]. When the parameters of the driving signal are taken outside the synchronization zone, the points of the intersections are scattered all over the chaotic attractor; see Fig. 15(b). Qualitatively similar transitions between the periodically ''synchronized'' and ''nonsynchronized'' chaotic oscillations are observed in the second example, which is presented in Figs. 15(c) and 15(d).

B. Chaotic and periodic orbits

Let us discuss a mechanism that explains this type of synchronization of chaotic oscillations. Since a chaotic attractor contains an infinite number of saddle periodic orbits, the motion of the system along the chaotic trajectory can be considered as switching between nearest saddle periodic orbits. Therefore, the set of unstable periodic orbits forms a ''skeleton'' of the chaotic attractor and define time characteristics of the chaotic oscillations. External periodic forcing applied to the system can lock a sufficient number of these periodic orbits. When this ''skeleton'' becomes phase locked with the periodic forcing, the chaotic trajectory switches between phase locked saddle periodic orbits. A well-known example of a chaotic attractor that contains the periodic ''skeleton,'' whose phases are related to the phase of external periodic signal, is the chaotic attractor in the periodically driven Duffing's equation. In other words, the qualitative difference between the nonsynchronized and periodically ''synchronized'' chaotic oscillations is the same as between low-dimensional chaotic oscillations in an autonomous system and the chaotic oscillations appearing due to a periodic driving of a two-dimensional nonlinear system, which is not a self-excited oscillator.

In the type of synchronization considered, the periodic forcing is needed only for locking the phases of the periodic orbits, but not for suppression of the instability. Due to this fact, such synchronization does not require large forcing amplitude. The existence of certain threshold value of the amplitude for this type of synchronization is caused, mainly, by the spread of the main frequencies of the saddle periodic orbits embedded in the synchronizing attractor.

C. Chaos suppression

Inside the synchronization zone the phase locked attractor is subject to bifurcations that can lead to the onset of stable periodic motions. For example, when the amplitude of driving increases, the circuit demonstrates the inverse sequence of the period doubling bifurcations from the phase locked chaotic attractor shown in Fig. 15(c). These bifurcations lead to the formation of different stable periodic orbits. One of these periodic orbits is presented in Fig. 16(a). The onset of stable periodic oscillations in the chaotic system can be considered as a mechanism for chaos suppression. In our case, the regime of stable periodic oscillations is observed within a certain interval of the amplitude values A. When the driving becomes too strong the periodic motion loses the stability and the system switches to unsynchronized chaotic behavior. The cross section of the attractor that corresponds to this nonsynchronized behavior is shown in Fig. 16(b).

The suppression of chaos in dynamical systems by means of external periodic forcing is a well-known phenomenon, see, for example, Refs. 31–35. Usually the suppression of chaos is provided by the onset of stable periodic oscillations. The possibility of the appearance of stable periodic motions in the phase space of the driven system is not surprising. Indeed, any static or periodic influence applied to a nonlinear system changes the parameters of the system. When parameters of a chaotic system change the system goes through various bifurcations. In many cases these bifurcation sequences are characterized by alternation of chaotic and periodic regimes of oscillations. By selecting the parameters of external forcing inside a periodic window, one can suppress chaos in the system.

V. THRESHOLD SYNCHRONIZATION

In this paper we have illustrated only the regimes of synchronized chaos that we observed in the coupled oscillators of smooth waveforms. It is clear that these regimes do not capture all possible forms of synchronized chaos. For instance, a different form of synchronized chaotic oscillations is observed in coupled relaxation oscillators. This type of chaos synchronization has a more transparent mechanism, and its dynamical features are different from the cases considered here. The nonlinear theory of synchronization between periodic relaxation oscillators was intensively studied in applications to electronic systems[36–38] and in biological systems; see, for example, Refs. 39 and 40 and references therein. The ability of relaxation oscillators to demonstrate individual chaotic dynamics was shown in many papers; see, for example, Refs. 41 and 42. The theoretical studies of the onset of synchronized chaotic pulsations and experimental observations of synchronization between chaotic relaxation circuits can be found in the papers.[43,44] It was experimentally demonstrated in the paper[44] that the synchronization and suppression of chaos in relaxation oscillators can be provided by external pulses. It was shown that there are chaotic relaxation oscillators that can, theoretically, be synchronized by a sequence of external pulses of arbitrary small amplitude. Of course, this sequence of pulses has to be perfectly matched to a chaotic or an unstable periodic regimes of pulsations in the unperturbed oscillator. This indicates the difference in features of the chaos synchronization between this class of relaxation oscillators and the oscillators of smooth waveforms.

VI. CONCLUSIONS AND OUTLOOK

Using the results of experimental observations, we have discussed a few forms of synchronized chaotic behavior. Despite the fact that we did not review all studied cases of synchronized chaotic behavior, we hope that the images of synchronized chaos presented in this paper give an idea of the variety of different forms of chaos synchronization.

Since the general framework for the phenomenon of synchronization of chaotic oscillations is not available, we restricted our considerations to the case of identical chaotic oscillations, and the cases of synchronized chaos in drive-response systems. In the case of identical chaotic oscillations, the synchronized chaotic attractor has a rather simple image. In the phase space of the coupled systems, the trajectories of this attractor are located on the integral manifold, which has the form of a hyperplane.

The cases of drive-response systems are simplified in the sense that the motions on the synchronized chaotic attractor are related to the motions on the attractor in the phase space of the driving system. To study the onset of chaos synchronization in these systems we use the fact of stable predictability of the response behavior from the driving chaotic signal. Using the auxiliary system as the predicting device, we reduce the problem of the investigation of complex synchronized trajectories in the phase space of the drive-response system to the analyses of stability and robustness of identical oscillations in response-auxiliary systems.

Analysis of chaotic signals measured from synchronized systems indicates phase locking between saddle periodic orbits embedded in the chaotic attractor in the phase space of the driving system, and the saddle periodic orbits in the synchronized attractor in the phase space of the response system. It seems plausible that such phase locking between the pairs of saddle periodic orbits in the coupled system is a bifurcation mechanism for the onset of synchronized chaotic oscillations. We believe that the idea of phase locking between saddle periodic orbits can be useful for building a general framework for synchronized chaotic oscillations.

In all examples of synchronized chaos considered in this paper, the interacting systems are physically separated. It is obvious that the phenomenon of synchronization can take place in the high-dimensional systems in which this separation is not clear. There are many known examples where the low-dimensional chaotic attractor appears in a system with high-dimensional phase space. This low-dimensional behavior of the system can also be explained by synchronization between different portions of the system. Due to this synchronization a small portion of the system can be considered as a driving subsystem, while the other part of the system operates as a synchronized response subsystem. It is clear that the partition of the system into the synchronized subsystems may be a very difficult problem. We believe that detailed studies of typical routes of the onset of synchronized chaos will give a background not only for detection of chaos synchronization but also for the partitioning of high-dimensional systems.

ACKNOWLEDGMENTS

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APPENDIX A: MECHANISMS FOR SYNCHRONIZATION OF PERIODIC OSCILLATIONS

In this appendix we discuss basic principles of synchronization of periodic oscillators by external periodic signals.

1. Threshold synchronization

At first, we consider the mechanism for synchronization in relaxation oscillators. Being a dissipative system, a relaxation oscillator contains a positive feedback or negative resistance that destabilizes static states. It also has a nonlinear element and a time defined element. The nonlinear element defines a threshold level and determines the amplitude of generated pulsations. The time interval between the pulsations is controlled by the time defined element. This element sets the time that is required by the system to recover from previous firing and to become ready for the next firing. During the recovery phase the system ''slowly'' approaches a threshold level where it fires by itself; see Fig. 17(a). Immediately before the threshold state the system is extremely sensitive to small external perturbations, which may cause the firing. Therefore, if small external pulses appear at the times when they are able to push the system through the threshold level, the firings will be in synchrony with the pulses; see Fig. 17(b). In addition to this, these external pulses change the period of the relaxation oscillations by shortening the process of slow evolution. It is clear that the synchronization with external pulses of small amplitude can

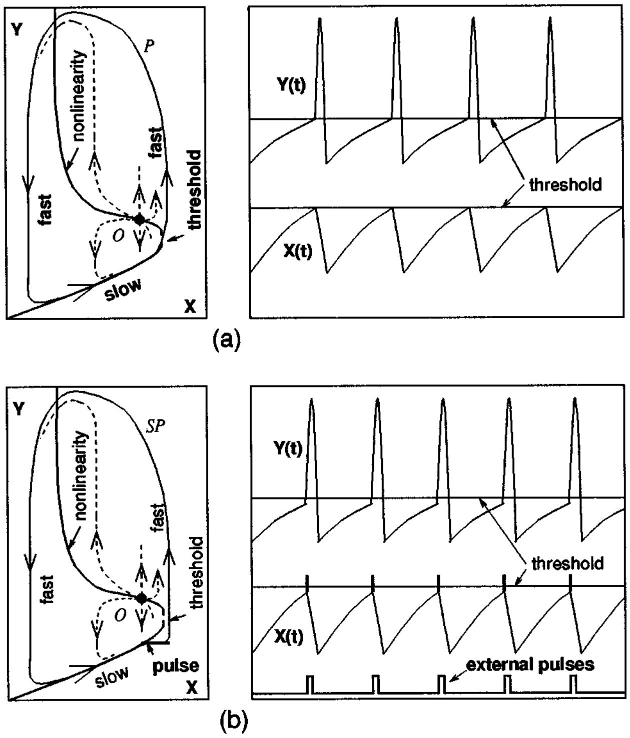

FIG. 17. Sketches of phase portraits (left) and waveforms (right) of an autonomous (a) and a synchronized (b) periodic relaxation oscillator. (a) Fast and slow motions along the stable limit cycle P correspond to the firing and recovering, respectively. Transitions from slow to fast motions occur at the threshold point. (b) The external pulses interrupt the slow motions and induce firing. When the period of the external pulses is a little less than the time interval of slow motions, and the amplitude of the pulses is sufficient to push the system through the threshold, then the external pulses synchronize the firings. SP is the image of the synchronized periodic oscillations.

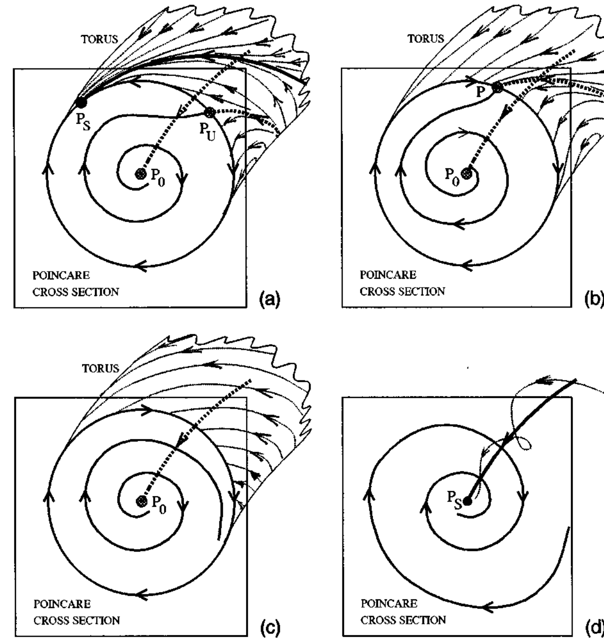

FIG. 18. Fragments of tori, periodic motions, and their Poincaré cross sections illustrating two mechanisms for the onset of synchronized periodic oscillators. The trajectories on the Poincaré cross sections are sketched out by thick curves. Synchronized periodic motion is marked by P_S. (a), (b), and (c) illustrate the transition from synchronized to unsynchronized oscillations in a system with weak periodic driving. The transition from synchronized oscillations that correspond to stable periodic motion P_S on the torus (a) to the quasiperiodic motions (c) due to the tangential bifurcation associated with stable P_S and unstable P_U periodic motions. These periodic motions merge with each other (b) and then disappear (c). The onset of synchronization caused by strong periodic driving is illustrated by the transition from (c) to (d). The torus merges into the unstable periodic motion P_0 (c) and as a result, the periodic motion becomes stable (d).

be achieved only if the period of these pulses is related to the duration of the recovering process. However, mention should be made that, in some cases, the external driving can significantly influence the initial conditions and/or the dynamics of the recovering process.

2. Synchronization and beats

Synchronization in generators of smooth waveforms mainly results from the action of dissipative forces. To illustrate the synchronization of periodic oscillations, we will use a simplified description of the synchronization mechanism. Being a dissipative system, any generator of periodic oscillations contains a positive feedback or negative resistance that destabilizes stationary states, and a nonlinear element that controls the amplitude of the oscillations. Unlike the relaxation oscillator, in the generator of smooth waveforms, the frequency of the oscillations is usually defined by a resonant element. Due to dissipation, the resonant element resonates within a certain frequency band. The nonlinearity and dissipation in the system select a particular form of periodic oscillations (mode). The stability of this mode indicates that this mode is preferred energetically and, therefore, the mode corresponds to a minimum of the energy functional. Due to the existence of a band of resonant frequencies, the characteristic frequencies of this mode change with a small variation of the parameters of the oscillator.

An external periodic forcing applied to the oscillator will periodically modulate the ''environment'' in which the intrinsic mode of oscillations lives. As a result of this modulation, the nonlinearity and dissipation will again tend to change the intrinsic mode of oscillations to a mode that is preferred energetically. A significant role in this process can by played by resonance between the perturbations and the mode. Therefore, the frequencies of the mode tend to become related to the frequencies of external forcing. If the perturbations introduced by the external forcing are not sufficient for the proper change of the intrinsic mode, then the system will be out of synchronization and will demonstrate quasiperiodic oscillations that are known as beats. In the phase space of the system these quasiperiodic oscillations correspond to the unlocked trajectories on the stable torus; see Fig. 18(c).

A. Weak forcing

There are two well-known mechanisms that describe change of the intrinsic mode of oscillations. One of these mechanisms is self-tuning of the frequencies of the unperturbed intrinsic mode. This self-tuning is caused by dissipative forces, therefore it is directed toward one of the minima of the energy functional. Being an integral characteristic this functional depends upon the phase relation between the mode and the perturbations induced by the external signal. As a result of it, this dependence has a periodic structure. Due to this periodicity, a stable mode of locked frequencies always appears together with an unstable mode of locked frequencies. In the phase space of the oscillator, this case corresponds to the formation of stable and unstable periodic motions on a torus; see Figs. 18(a)–18(c). This mechanism is typical for external forcing with small amplitude. In order to achieve synchronization with small amplitude, the frequencies induced by the forcing have to be very close to resonance with the frequencies of the unperturbed mode of oscillations. It is clear that the amplitude of perturbations required for locking depends on how much the spectrum of the inner mode has to be changed in order to provide a resonance with the perturbations induced by the external forcing. Therefore, when the frequencies of the perturbations are in resonance with frequencies of an unperturbed intrinsic mode, then the amplitude of the external forcing that is sufficient for locking can be infinitely small. Of course, this is correct only for the synchronization of stable periodic oscillations in systems without noise.

B. Strong forcing

The second mechanism for the change of the intrinsic mode is the competition between the inner and induced modes excited in the oscillator. Due to nonlinearity and dissipation one mode of oscillations causes strong dissipation for the other modes. As a result of this competition the strongest mode will tend to eliminate the other modes. This mechanism is observed for forcing of large amplitude. In this case the independent intrinsic mode simply disappears. This mechanism can also be distinguished from the viewpoint of bifurcation theory. It is characterized by the merging of the torus into unstable periodic motion; see Figs. 18(c) and 18(d). As the result of this bifurcation, the torus disappears and the periodic motion becomes stable.

APPENDIX B: IMPLEMENTATION OF THE CIRCUIT

In order to study chaos synchronization in experiments, we employ electronic circuits whose individual dynamics are known to be chaotic. The chaotic circuit consists of a nonlinear converter N and linear feedback; see the diagram of the driving circuit in Fig. 1. Chaotic circuits of such architecture was studied by Dmitriev *et al.*[45] In our experiments we used a nonlinear converter whose diagram is shown in Fig. 19. The converter transforms input voltage x into an output, which has nonlinear dependence $\alpha f(x)$ on the input. The parameter α stands for the gain of the converter at $x=0$. The shape of the nonlinearity produced by the converter, N, is shown in Fig. 20.

The linear feedback contains the low pass filter (RC') and the resonant circuit rLC. Depending on the parameters of the linear feedback and the parameter of the nonlinearity,

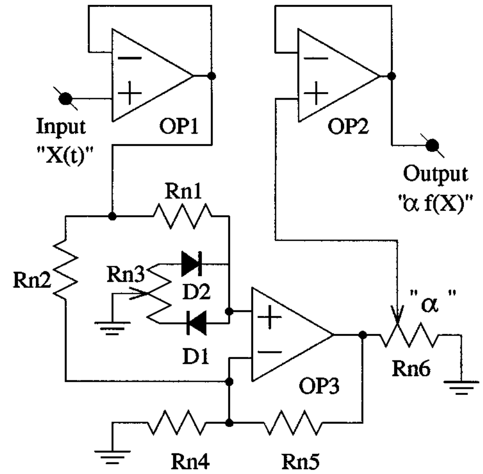

FIG. 19. The diagram of the nonlinear converter. $Rn1=2.7$ kΩ, $Rn2=Rn4=7.4$ kΩ, $Rn3=100$ Ω, $Rn5=186$ kΩ, $Rn6=2$ kΩ. The diodes D1 and D2 are of the 1N4148 type. The operational amplifiers OP1 and OP2 are both the TL082 type and the operational amplifier OP3 is of the LF356N type.

α, the circuit can generate various regimes of periodic and chaotic oscillations. It can be easily shown that dynamics of the circuit is described by the following system of differential equations:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - \delta x_2 + x_3, \\ \dot{x}_3 &= \gamma[\alpha f(x_1) - x_3] - \sigma x_2. \end{aligned} \tag{B1}$$

$x_1(t)$ is the voltage across the capacitor C, $x_2(t)=\sqrt{L/C}J(t)$, with $J(t)$ the current through the inductor L. $x_3(t)$ is the voltage across the capacitor C'. Time has been scaled by $1/\sqrt{LC}$. The parameters of this system have the following dependence on the physical values of the circuit elements: $\gamma=\sqrt{LC}/RC'$, $\delta=r\sqrt{C/L}$ and $\sigma=C/C'$.

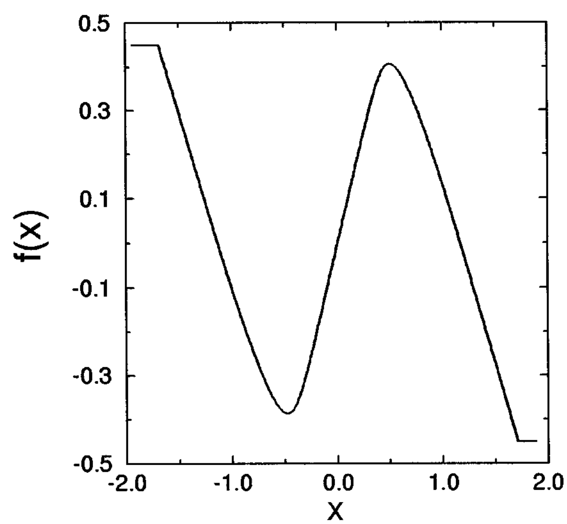

FIG. 20. The shape of the nonlinear function $f(x)$ measured in the experiment.

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