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Threshold synchronization of chaotic relaxation oscillations

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A new type of synchronized chaos conditioned by the threshold synchronization of relaxation oscillators with *chaotic* behaviour is studied experimentally. It is shown that for a certain parameter ratio the pulses generated by the chaotic oscillator may be synchronized by periodic and chaotic pulse sequences generated by the drive oscillator. It is found that the threshold synchronization regime enables one to detect modulation of chaotic signals.

1. The synchronization phenomenon in systems with chaotic behaviour is known to lead to different nonlinear effects. It was shown in refs. [1–5] that small time-dependent perturbations of a parameter of a system with chaotic behaviour can eliminate chaos as a result of stabilization of periodic motion near the unstable limit cycle which is contained in the chaotic attractor. It is also known that in the case of coupled systems with individual chaotic behaviour the transition to synchronization may give rise to a regime of synchronized chaotic oscillations [6–14]. It is obvious that like in the case of synchronization of periodic oscillations these phenomena may play a significant role in the behaviour of many physical systems as well as in different applications.

Although synchronization in different chaotic systems was described in quite a few papers, the features of this phenomenon in *chaotic* relaxation systems still need further investigation. The alternation of fast motion (pulsation) and slow evolution is typical for relaxation systems. The transition from slow to fast motion occurs at the moments when the system reaches a certain threshold state. Immediately before the threshold the relaxation system is extremely sensitive to small external perturbations which may cause a forced transition to fast motion. This property is widely used in electronics for synchronization of periodic pulse oscillators driven by weak periodic synchronization signals. This mechanism of synchronization is known as threshold synchronization [15]. A similar type of synchronization phenomenon was considered for the problem of synchronization of pulse-coupled biological periodic oscillators [16,17].

This paper presents some results of theoretical and experimental investigation of threshold synchronization in relaxation systems obtained by modeling with special electronic circuits.

The paper includes a brief description of the electronic circuit used for modeling the relaxation chaotic system (section 2), investigation of the effect of chaos suppression in this system by applying weak periodic pulses (section 3), and synchronization of chaos in a pair of unidirectionally coupled relaxation chaotic systems (section 4). Examples of applications of synchronized chaotic relaxation oscillators are discussed in section 5.

2. The algorithm for the performance of a chaotic relaxation circuit can be realized within a block-scheme shown in fig. 1a. Slow evolution of the system corresponds to an increasing voltage u_2 on the capacitor C when charged. When this voltage reaches the threshold level, $u_2(t)=0$, the comparator switches to another state and the pulse oscillator generates a pulse d_n at time t_n, a sample circuit controlled by this pulse provides recharging of the capacitor C to the voltage $u_2=-\sigma u_1(t_n)$ and sets the nonlinear curve generator to the initial state $u_1=0$. After that, the capacitor C is charged from a current source I_0 and,

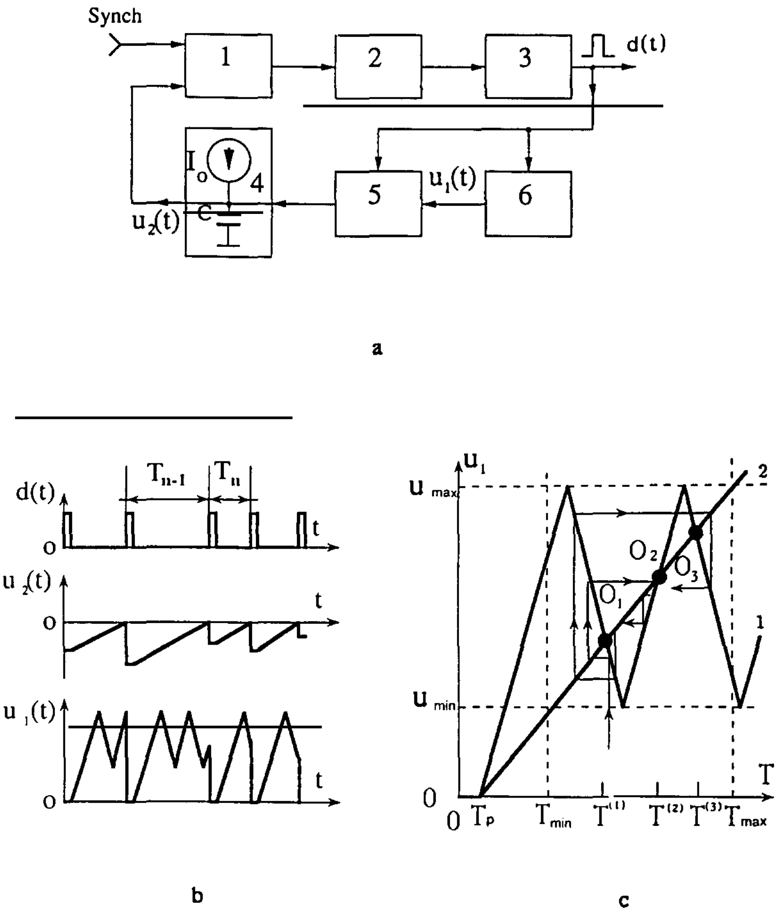

Fig. 1. (a) Block-scheme of the chaotic relaxation oscillator. (1) Adder, (2) comparator (threshold device), (3) pulse generator, (4) time defined circuit, (5) sample circuit, (6) nonlinear curve generator. (b) Possible waveforms $d(t)$, $u_2(t)$ and $u_1(t)$. (c) A possible map (1) with a triangular shaped function $F(T)$. (1) Function $F(T)$, (2) the dependence $u_1=T/\alpha$. A trajectory is marked by the arrow.

when its voltage, u_2, reaches the threshold level again, the pulse oscillator generates the next pulse, d_{n+1}. So the circuit produces a pulse sequence $d(t)$ with time intervals between the pulses determined as

$$T_{n+1}=\alpha u_1(t_n)+T_p, \quad u_1(t_n)=F(T_n) , \qquad (1)$$

where $\alpha=\sigma C/I_0$, $F(T)$ is the output voltage of a nonlinear curve generator. We use a triangular type function $F(t)$ (figs. 1b and 1c) with the parameters $u_{min}=2$ V and $u_{max}=7$ V. The slopes of the function $F(T)$ on increasing and decreasing intervals were assumed to be the same and equal to $\beta=|\beta'|$, where $\beta'=dF/dT$. α and β were used as control parameters.

A model of a chaotic relaxation oscillator (1) is a 1D map. The trajectories of this map can be easily studied by means of the plot presented in fig. 1c. If the parameters α and β are chosen so that $\alpha\beta>1$ all trajectories of the map come to a strange attractor that is bounded by $T_{min}=\alpha u_{min}$ and $T_{max}=\alpha u_{max}$ and characterized by a positive Lyapunov exponent $\lambda=\ln(\alpha\beta)>0$. Choosing $F(T)$ as a piecewise linear function one can design an oscillator which provides a smooth variation of the characteristics of the chaotic oscillations in a broad parameter region.

3. A strange attractor mapped by (1) contains one or more unstable fixed points O_i which are located at the intersections of curves 1 and 2 (see fig. 1c). The points correspond to the existence of the regime of periodic oscillations of the oscillator in the form

of a pulse sequence with period $T^{(i)}$. It is obvious that such regimes are unstable due to the instability of the fixed points.

Consider the weak periodic forcing of a chaotic oscillator. In order to study this influence in an experiment, an external periodic pulse signal $d_{dr}(t)$ with amplitude E and period T_{dr} is applied to the input "Synch". The pulses d_{dr} are added to the voltage $u_2(t)$. If at the moment of the pulse action the voltage u_2 is close enough to the threshold level, i.e. the conditions $0>u_2>-E$ are satisfied, then the electronic circuit will generate a pulse d in step with d_{dr}. Otherwise the pulse d_{dr} has no influence on the operation of the circuit.

In the case of small amplitude E ($E \ll T_{max}/\alpha$), the signal d_{dr} provides synchronization if the parameters of the signal satisfy the following conditions,

$$\alpha E/\sigma > (T_{dr} - T^{(i)})(\alpha\beta^{(i)} - 1) > 0 , \quad (2)$$

where

$$\beta^{(i)} = \left.\frac{dF}{dT}\right|_{T=T^{(i)}} .$$

Figure 2 shows the threshold synchronization zones in the plane of the parameters (T_{dr}, E) when the parameters of the oscillator are like in fig. 1c. One can see from fig. 2 that even for very weak periodic forcing chaos can be suppressed and a regime of periodic synchronized oscillations will be realized. Mention should be made that threshold synchronization is possible if the period of weak forcing is close enough to the period $T^{(i)}$.

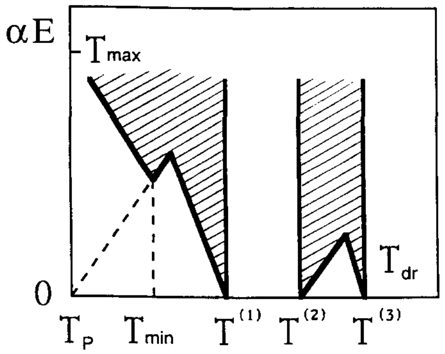

Fig. 2. Synchronization zones of the chaotic oscillator forced by a periodic pulse signal.

Note that, besides the unstable fixed points O_i, map (1) contains an infinite number of unstable limit cycles. These periodic motions can also be synchronized by periodic forcing at higher order synchronisms. This indicates the existence of a complex fractal structure of the synchronization zones for periodic oscillations of different types. This complex structure is not shown in fig. 2.

4. It should be emphasized that the solutions of map (1) include open unstable trajectories which belong to the strange attractor. These motions correspond to the oscillatory regime of the system in which the output signal $d(t)$ looks like a chaotic sequence of pulses. It seems interesting to investigate the threshold synchronization along one of these trajectories if the synchronizing signal is generated by the same chaotic oscillator.

In an experimental investigation of threshold synchronization by external chaotic forcing, the output pulses d_{dr} with amplitude E generated by a driving chaotic oscillator were led to the input "Synch" (see fig. 1a) of the synchronizing chaotic oscillator. The parameters u_{min}, u_{max} and T_p of the driving and synchronizing oscillators were set to be close and fixed. The parameters α_{dr}, β_{dr}, α_s, β_s and the normalized amplitude $\epsilon = E/\sigma u_{max}$ of the pulses d_{dr} were taken as control parameters. Here and in the following the subscripts "dr" and "s" denote the parameters of driving and synchronizing oscillators, respectively.

The synchronization regions in the parameter plane (α_s, ϵ) with fixed values of α_{dr} and $\beta_{dr} = \beta_s$ are shown in fig. 3a. The diagrams in fig. 3a show that locking is possible if $\alpha_s > \alpha_{dr}$ which indicates that, for threshold synchronization, each synchronizing pulse d_{dr} should appear before the voltage u_{2s} reaches the threshold level. The loss of synchronization with increasing α_s is caused by the fact that the voltage u_{2s} is so small that, at the moment the pulse d_{dr} appears, the threshold level cannot be reached. It is quite easy to show that if the parameters u_{max}, u_{min}, T_p and β are the same in both oscillators, the synchronization zone is defined by the following conditions,

$$\alpha_s > \alpha_{dr}, \quad \epsilon > 1 - \alpha_{dr}/\alpha_s . \quad (3)$$

The spread in the parameters leads to narrowing of the synchronization zones which vanish when $\epsilon = \epsilon_{min} > 0$.

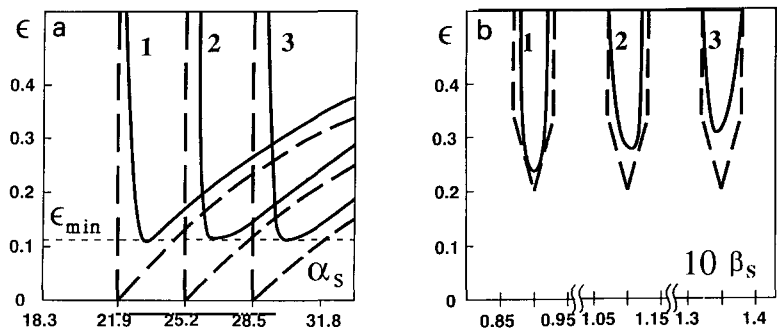

Fig. 3. Parameter regions corresponding to regimes of synchronized chaotic oscillations. (a) Parameter plane (α_s, ϵ) with $\beta_{dr}=\beta_s=0.09$ V/μs. Zones 1, 2 and 3 correspond to α_{dr} (μs/V) equal to 21.9, 25.2 and 28.5. (b) Parameter plane (β_s, ϵ) with $\alpha_{dr}=25.2$ μs/V and $\alpha_s=31.8$ μs/V. Zones 1, 2 and 3 correspond to β_{dr} (V/μs) equal to 0.09, 0.11 and 0.135. The theoretical boundary is marked by the dashed line. The solid line shows the boundary found in experiment.

Note that, unlike the periodic oscillations for which only one parameter (period) is essential, the synchronization of chaotic oscillations is determined by a few parameters. Consequently the threshold synchronization of chaotic signals is possible only when all parameters of the oscillators are sufficiently close. For example β can be considered as an additional control parameter. Analysis of map (1) shows that when the correspondent parameters u_{min}, u_{max} and T_p are equal but $\beta_{dr}\neq\beta_s$, synchronization can be realized if

$$\epsilon > 1-\alpha_{dr}/\alpha_s=\epsilon_{cr}, \quad |\beta_s-\beta_{dr}|<(\epsilon-\epsilon_{cr})/\alpha_s\,,$$
$$|\beta_s-\beta_{dr}|<u_{min}\epsilon_{cr}/u_{max}\alpha_{dr}\,. \qquad (4)$$

The region of synchronization in the parameter plane (β_s, ϵ) is presented in fig. 3b for some values of β_{dr} and fixed α_s and α_{dr}. One can see that locking occurs only in a narrow zone of β_s close to β_{dr}.

5. The threshold synchronization mechanism is quite general for self-oscillatory relaxation systems. For example, there are many studies devoted to synchronization of periodic relaxation oscillations by external periodic stimulation in biological systems (excitable membranes, cardiac models, breathing systems, etc.) [18–21]. It is known (see, for instance, refs. [19,22]) that under certain conditions the behaviour of biological systems becomes chaotic. Therefore it seems natural to suppose that the effects related to threshold synchronization are also intrinsic to the dynamics of biological chaotic oscillators.

There are a number of works [23–25] in which chains of coupled relaxation oscillators are used as earthquake fault models. The dynamics of these models leads to the appearance of spatio-temporal chaos caused by irregular formation and disintegration of domains which consist of several neighbouring oscillators near the threshold state. If the model takes into account the interaction between the domains conditioned by sound wave generation, then the threshold synchronization regime can provide simultaneous generation of pulsations at different points in space.

Different ways of employing systems with chaotic dynamics in communication systems for secret information were suggested in a number of recent papers [26–28]. The threshold synchronization seems also to be useful in realizing this idea. As it was mentioned above, the pulse oscillator described in section 2 enables one to change the characteristics of a chaotic signal smoothly in a broad parameter region. This property can be used for modulation, the regime of threshold synchronization for demodulation of chaotic signals.

It can be seen that the voltage u_{2s} never reaches the threshold level if the oscillator operates in the regime of threshold synchronization. The mean value of this voltage at the moment a synchronizing pulse appears, $\bar{u}$, depends on the parameter difference (see fig. 4a). This dependence has a linear part where the voltage $\bar{u}$ is proportional to α_{dr}. This property can be used for demodulation. The process of demodulation is shown in fig. 4b for the sinusoidally modu-

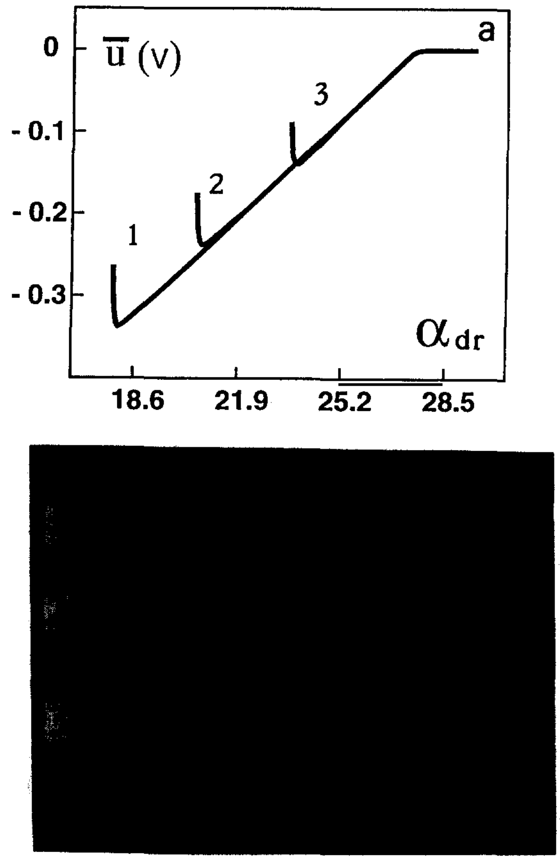

Fig. 4. (a) Voltage $\bar{u}$ versus the parameter α_{dr} with $\beta_{dr}=\beta_s=0.09$ V/μs and $\alpha_s=28.5$ μs/V. Lines 1, 2 and 3 are obtained for E equal to 0.4, 0.3 and 0.15 V, respectively. (b) Waveforms in the regime of synchronized chaos with parameter modulation of the driving oscillator: (1) $\alpha_{dr}(t)=\alpha_0[1+m\sin(\Omega t)]$, (2) $u_{2dr}(t)$ and (3) $u_{2s}(t)$.

lated parameter α_{dr} of a driving oscillator $\alpha_{dr}(t)=\alpha_0[1+m\sin(\Omega t)]$. It can be seen from the diagram that the banding line of the voltage $u_{2s}(t)$ is a copy of the modulation signal $\alpha_{dr}(t)$. It should be emphasized that the regime of threshold synchronization and, consequently, the access to information is possible only if all parameters of the transmitting and receiving systems (oscillators) are close.

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